

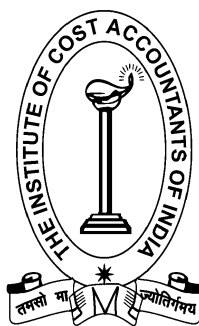
**FOUNDATION**

**Paper 3**

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# **Fundamentals of Business Mathematics and Statistics**

**Study Notes**  
**SYLLABUS 2022**



**The Institute of Cost Accountants of India**

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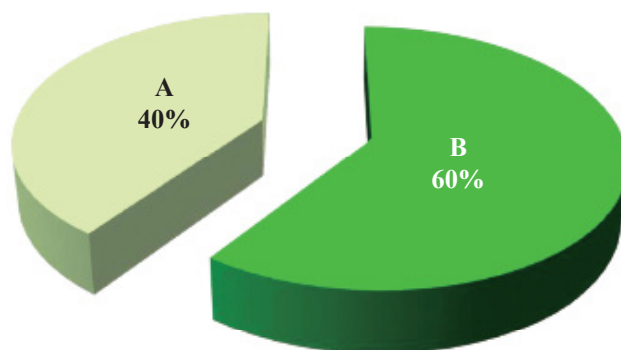
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# PAPER 3: FUNDAMENTALS OF BUSINESS MATHEMATICS AND STATISTICS

## Syllabus Structure:

The syllabus in this paper comprises the following topics and study weightage:

Module No.	Module Description	Weight
<b>Section A: Fundamentals of Business Mathematics</b>		<b>40%</b>
1	Arithmetic	15%
2	Algebra	20%
3	Calculus - Application in Business	5%
<b>Section B: Fundamentals of Business Statistics</b>		<b>60%</b>
4	Statistical Representation of Data	5%
5	Measures of Central Tendency and Dispersion	15%
6	Correlation and Regression	15%
7	Probability	15%
8	Index Numbers and Time Series	10%





# Contents as per Syllabus

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- 1.2 Time Value of Money and Annuity – Simple and Compound Interest
- 1.3 Arithmetic Progression and Geometric Progression
- 1.4 Time and Distance

### **Module 2 : Algebra** 89 - 188

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- 2.2 Indices and Logarithms (Basic Concepts)
- 2.3 Permutation and Combinations (Basic Concepts)
- 2.4 Quadratic Equation (Basic Concepts)

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**SECTION - A**

**FUNDAMENTALS OF BUSINESS MATHEMATICS**



# ARITHMETIC

1

**This Module includes:**

- 1.1 Ratios, Variations and Proportions**
- 1.2 Time Value of Money and Annuity – Simple and Compound Interest**
- 1.3 Arithmetic Progression and Geometric Progression**
- 1.4 Time and Distance**

# ARITHMETIC

## Module Learning Objectives:

Upon completion of study this module, students will be able to understand –

- ✦ The concepts of Ratios and Proportions as well as their different compositions
- ✦ The different pattern of Variations among variables
- ✦ The application areas of Ratios, Proportions and Variations in real life situations
- ✦ What is actually meant by Time value of Money
- ✦ The concept of Simple and Compound interest
- ✦ What is meant by Annuity
- ✦ How to calculate Simple and Compound interests and Annuities under different time scales
- ✦ How to apply time value of money practical situations
- ✦ What is actually meant by Arithmetic & Geometric progression
- ✦ The properties of AP and GP
- ✦ The relation between time and distance and application of the same

# Ratios, Variations and Proportions

## 1.1

### Ratios:

Ratio is the comparative relation between two quantities of same kind expressed in the same units.

**Example:** In a class test  $A$  secured 80 marks and  $B$  secured 40 marks out of 100 then we can compare that  $A$  secures double that of  $B$ .

i.e. Ratio is  $80/40 = 2$ , is a pure natural number, and no unit is associated with it.

### Note:

1. In the ratio  $a:b$ ,  $a$  &  $b$  are called the terms of the ratio. Here ' $a$ ' is called Antecedent and ' $b$ ' is called Consequent.

### Properties of Ratio:

1. The value of ratio remains unchanged when the terms of the ratio are multiplied or divided by the same number.

Ex: 2:4, Multiplied by 2, 4:8;                      9:27, divided by 9, 1:3

2. Two or more ratios can be compared by reducing them to the same denominator.

Ex: In the ratios 3:4 and 8:12 which is greater?

*multiplying the terms of 3:4 by 3, we get 9:12*

*multiplying the terms of 8:12 by 1, we get 8:12*

As  $9:12 > 8:12$ , we conclude  $3:4 > 8:12$

### Ratio of equality and inequality:

1. If  $a = b$  then the ratio  $a:b$  is called equal ratio.

Ex: 3:3, 4:4 etc. are equal ratios.

2. If  $a > b$  then the ratio  $a:b$  is called greater inequality.

Ex: 4:3, 9:7 etc. are the ratios of greater inequalities.

3. If  $a < b$  then the ratio  $a:b$  is called lesser inequality.

Ex: 3:4, 7:9 etc. are the ratios of lesser inequalities.

**Inverse ratio or reciprocal ratio:**

The inverse ratio of  $a:b$  is  $b:a$

**Different kinds of ratios:****1. Compound ratio:**

If two or more ratios are multiplied together then the resultant ratio is called Compounded ratio.

**Ex:** For the ratios  $a:b, c:d, e:f$  the compounded ratio is  $ace:dbf$

**Note:**

The compounded ratio of two reciprocal ratios is unity. i.e.  $a:b$  is the reciprocal ratio of  $b:a$

Then the compounded ratio is  $ab:ba = 1$

**2. Duplicate ratio:**

If two equal ratios are compounded together then the resulting ratio is called Duplicate ratio of the given ratio i.e. the duplicate ratio of  $a:b$  is  $a:b \times a:b = a^2:b^2$ .

**3. Triplicate ratio:**

If three equal ratios are compounded together then the resulting ratio is called Triplicate ratio of the given ratio i.e. the triplicate ratio of  $a:b$  is  $a:b \times a:b \times a:b = a^3:b^3$ .

**4. Sub duplicate ratio and Sub triplicate ratios:**

$\sqrt{a}:\sqrt{b}$  is the Sub Duplicate ratio of  $a:b$  and  $\sqrt[3]{a}:\sqrt[3]{b}$  is the Sub Triplicate ratio of  $a:b$

**5. Continued ratio:**

The continued ratio is the relation between the magnitudes of two or more ratios and is denoted by  $a:b:c$

**Ex:** The continued ratio of 2:3 and 4:10 is 8:12:30 or, 4:6:15

**Points to remember:**

1. Bring the quantities to the same units while finding a Ratio.  
Ex: if  $A = ₹2$  and  $B = 50$  p  
Then  $A:B = 200:50 = 4:1$
2. When the quantity is increased by given ratio multiply the quantity by greater ratio.
3. When the quantity is decreased by given ratio multiply the quantity by lesser ratio.
4. When both increasing and decreasing of quantities are present in a problem multiply the quantity by greater ratio to increase and multiply the result by lesser ratio to obtain the final result.

### Proportions (OR) proportional:

If two ratios are equal then we say that the two ratios are in proportion. In other words the four quantities  $a, b, c$  and  $d$  are said to be in proportion if  $a:b = c:d$  Or,  $\frac{a}{b} = \frac{c}{d}$

Here the first and last quantities i.e.  $a$  &  $d$  are extremes and the two middle terms  $b$  &  $c$  are called means.

#### Property:

The four quantities  $a, b, c, d$  are in proportion if  $\frac{a}{b} = \frac{c}{d}$  Or,  $ad = bc$ . Converse is also true.

### Continued proportion:

If 3 quantities  $a, b$ , and  $c$  are such that  $a:b = b:c$  then we say that these 3 quantities are in continued proportion. If 3 quantities are in continued proportion then we get  $b^2 = ac$ .

**Ex:** 3, 6, 12 are in continued proportion.

$$\text{So } 6^2 = 3 \times 12 = 36$$

**Ex:** 2, -4, 8 are in continued proportion

$$\text{So } (-4)^2 = 2 \times 8 = 16$$

### Mean Proportional, Third Proportional and Fourth Proportional:

If  $x, y, z$  are in continued proportion then  $y$  is called the Mean Proportional (or Geometric Mean) of  $x$  &  $z$ . Thus  $\frac{x}{y} = \frac{y}{z}$  or,  $y^2 = xz$  or  $y = \sqrt{xz}$

This is the formula to obtain Mean Proportional ( $y$ ) of the numbers  $x$  &  $z$

Here  $z$  is called the Third Proportional of  $x$  and  $y$ . Thus the same formula as above can be used to find at 3rd proportional of  $x$  &  $y$ .

Similarly if  $a, b, c$  and  $d$  are in proportion i.e.  $a:b = c:d$  then  $d$  is called the Fourth Proportional of  $a, b$  and  $c$ .

#### Example:

(i) Mean proportional between 0.08 and 0.18 is  $\sqrt{0.08 \times 0.18} = \sqrt{.0144} = 0.12$

(ii) Third Proportional to 16 & 36 is obtained as  $16:36 :: 36:x$ , where  $x$  is the 3rd Proportional. So  $\frac{16}{36} = \frac{36}{x}$  or  $16x = 36 \times 36$  or,  $x = \frac{36 \times 36}{16} = 81$ . Hence 81 is the 3rd Proportional to 16 & 36

(iii) Fourth Proportional to 4, 9 and 12 is calculated as  $4:9 :: 12:X$ , where  $X$  is the 4th Proportional.

So  $4x = 9 \times 12$  or,  $X = 27$ . Hence 27 is the 4th proportional to 4, 9 & 12.

### Basic rules of proportions:

#### 1. Invertendo:

If  $a:b = c:d$  which implies  $b:a = d:c$  then we say that the operation of Invertendo is applied on the given proportion.

**2. Alternendo:**

If  $a : b = c : d$  which implies  $a : c = b : d$  then we say that the operation of Alternendo is performed on the given proportion.

**3. Componendo:**

If  $a : b = c : d$  which implies  $a + b : b = c + d : d$  then we say that the operation of Componendo is performed on the given proportion.

**4. Dividendo:**

If  $a : b = c : d$  which implies  $a - b : b = c - d : d$  then we say that the operation of Dividendo is performed on the give proportion.

**5. Componendo and dividendo:**

If  $a : b = c : d$  which implies  $a + b : a - b = c + d : c - d$  then we say that the operation of Componendo and Dividendo has been performed on the given proportion.

**6. Important theorem:**

If  $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} \dots\dots\dots$  then each ratio  $= \left\{ \frac{pa^n + qc^n + re^n \dots\dots}{pb^n + qd^n + rf^n \dots\dots} \right\}^{\frac{1}{n}}$

Where  $p, q, r, \dots\dots$  are quantities

Proof:

Let  $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} \dots\dots = k$  (Say)

So  $\frac{a}{b} = k$ , Or  $a = bk$

Similarly  $\frac{c}{d} = k$ , Or,  $c = dk$  &  $\frac{e}{f} = k$ , Or  $e = fk \dots\dots$ . So on

RHS

$$= \left\{ \frac{pa^n + qc^n + re^n \dots\dots}{pb^n + qd^n + rf^n \dots\dots} \right\}^{\frac{1}{n}} = \left\{ \frac{p(bk)^n + q(dk)^n + r(fk)^n \dots\dots}{pb^n + qd^n + rf^n \dots\dots} \right\}^{\frac{1}{n}}$$

$$= \left\{ \frac{pb^n k^n + qd^n k^n + rf^n k^n \dots\dots}{pb^n + qd^n + rf^n \dots\dots} \right\}^{\frac{1}{n}}$$

$$= \left\{ k^n \left( \frac{pb^n + qd^n + rf^n \dots\dots}{pb^n + qd^n + rf^n \dots\dots} \right) \right\}^{\frac{1}{n}}$$

$$= \{k^n\}^{1/n} = k$$

**Note:**

1. Put  $n = 1$

$$\text{So } \frac{pa + qc + re + \dots}{pb + qd + rf + \dots} = \text{each ratio} = k$$

2. Put  $p = q = r = \dots = 1$

$$\text{So } \left\{ \frac{a^n + c^n + e^n + \dots}{b^n + d^n + f^n + \dots} \right\}^{1/n} = \text{each ratio}$$

3. The continued ratio

$$x : y : z = a : b : c$$

$$\text{Can be written as } \frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

4. If  $x : y = a : b$  it does not mean that  $x = a$  &  $y = b$

But  $x = ka, y = bk$ , where  $k$  is a constant

**Illustration 1.**

$$\text{If } \frac{4x - 3z}{4c} = \frac{4z - 3y}{3b} = \frac{4y - 3x}{2a}, \text{ show that each ratio} = \frac{x + y + z}{2a + 3b + 4c}$$

**Solution:**

$$\frac{4x - 3z}{4c} = \frac{4z - 3y}{3b} = \frac{4y - 3x}{2a} = \frac{4x - 3z + 4z - 3y + 4y - 3x}{4c + 3b + 2a} = \frac{x + y + z}{2a + 3b + 4c} \text{ (Proved)}$$

**Illustration 2.**

$$\text{If } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} \text{ Show } \frac{aceg}{bdfh} = \frac{a^4 + c^4 + e^4 + g^4}{b^4 + d^4 + f^4 + h^4}$$

**Solution:**

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = k \text{ (say)}$$

$$\text{So } a = bk, c = dk, e = fk, g = hk$$

$$\text{LHS} = \frac{aceg}{bdfh} = \frac{bk \times dk \times fk \times hk}{bdfh} = k^4$$

RHS

$$\frac{a^4 + c^4 + e^4 + g^4}{b^4 + d^4 + f^4 + h^4} = \frac{b^4 k^4 + d^4 k^4 + f^4 k^4 + h^4 k^4}{b^4 + d^4 + f^4 + h^4} = k^4$$

So LHS = RHS (Proved)

**Illustration 3.**

If  $a_1, a_2, \dots, a_n$  be continued proportion, show that  $\frac{a_1}{a_n} = \left(\frac{a_1}{a_2}\right)^{n-1}$

**Solution:**

$$\frac{a_1}{a_2} = \frac{a_2}{a_3} = \frac{a_3}{a_4} = \dots = \frac{a_{n-1}}{a_n} = \dots \text{.k(say)}$$

$$k^{n-1} = \frac{a_1}{a_2} \times \frac{a_2}{a_3} \times \frac{a_3}{a_4} = \dots \times \frac{a_{n-1}}{a_n} = \frac{a_1}{a_n}$$

Again  $k^{n-1} = \left(\frac{a_1}{a_2}\right)^{n-1}$

So  $\frac{a_1}{a_n} = \left(\frac{a_1}{a_2}\right)^{n-1}$  (Proved)

**Illustration 4.**

$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$  Prove that  $\frac{x^2 - yz}{a^2 - bc} = \frac{y^2 - zx}{b^2 - ca} = \frac{z^2 - xy}{c^2 - ab}$

**Solution:**

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k \text{ (Say)}$$

So  $x = ak, y = bk, z = ck$

$$\frac{x^2 - yz}{a^2 - bc} = \frac{(ak)^2 - bk \cdot ck}{a^2 - bc} = \frac{k^2(a^2 - bc)}{a^2 - bc} = k^2$$

Similarly  $\frac{y^2 - zx}{b^2 - ca} = \frac{z^2 - xy}{c^2 - ab} = k^2$

So  $\frac{x^2 - yz}{a^2 - bc} = \frac{y^2 - zx}{b^2 - ca} = \frac{z^2 - xy}{c^2 - ab}$  (Proved)

**Illustration 5.**

If  $\frac{p}{b-c} = \frac{q}{c-a} = \frac{r}{a-b}$  Prove that  $p + q + r = 0 = pa + qb + rc$

**Solution:**

$$\frac{p}{b-c} = \frac{q}{c-a} = \frac{r}{a-b} = k \text{ (say)}$$

So  $p = k(b-c), q = k(c-a), r = k(a-b)$

$$\text{So } p + q + r = k(b - c) + k(c - a) + k(a - b) = 0$$

$$\text{So } pa + qb + rc = k(b - c)a + k(c - a)b + k(a - b)c = 0$$

Hence the result (Proved)

### Illustration 6.

The marks obtained by four examinees are as follows: A: B = 2:3, B: C = 4:5, C: D = 7:9, find the continued ratio.

**Solution:**

$$A:B = 2:3$$

$$\text{So B: C} = 4:5 = 4 \times \frac{3}{4} : 5 \times \frac{3}{4} = 3 : \frac{15}{4}$$

$$\text{So C: D} = 7:9 = 7 \times \frac{15}{28} : 9 \times \frac{15}{28} = \frac{15}{4} : \frac{135}{28}$$

$$\text{So } A : B : C : D = 2 : 3 : \frac{15}{4} : \frac{135}{28} = 56 : 84 : 105 : 135$$

### Illustration 7.

Two numbers are in the ratio of 3:5 and if 10 be subtracted from each of them, the remainders are in the ratio of 1:5, find the numbers.

**Solution:**

$$\text{Let the numbers be } x \text{ and } y, \text{ so that } \frac{x}{y} = \frac{3}{5} \text{ Or, } 5x = 3y \dots \dots \dots (1)$$

$$\text{Again } \frac{x-10}{y-10} = \frac{1}{5}$$

$$\text{Or, } 5x - 50 = y - 10$$

$$\text{Or, } 3y - 50 = y - 10 \text{ [As } 5x = 3y \text{ from (1)]}$$

$$\text{Or, } 2y = 40, \text{ So } y = 20$$

$$\text{And } 5x = 3y = 60, \text{ So } x = 12 \text{ and } y = 20$$

### Illustration 8.

The ratio of annual incomes of A and B is 4:3 and their annual expenditure is 3:2. If each of them saves ₹1000 a year, find their annual income.

**Solution:**

Let the incomes be 4x and 3x (in ₹)

$$\frac{4x - 1000}{3x - 1000} = \frac{3}{2}$$

$$\text{From above } x = 1000$$

$$\text{So income of } A = ₹4,000 \text{ \& that of } B = ₹3,000$$

**Illustration 9.**

The prime cost of an article was three times the value of material used. The cost of raw materials was increased in the ratio 3:4 and the productive wage was increased in the ratio 4:5. Find the present prime cost of an article, which could formerly be made for ₹180.

**Solution:**

Prime cost =  $x + y$ , where  $x$  = productive wage,  $y$  = material used

Now prime cost = 180 =  $3y$  or,  $y = 60$ , again  $x + y = 180$ ,  $x = 180 - y = 180 - 60 = 120$

Present material cost =  $4y/3$ , present wage  $5x/4$

Present prime cost =  $4 \times 60/3 + 5 \times 120/4 = ₹230$

**Illustration 10.**

If  $\frac{3x+4}{4x+9}$  is the duplicate ratio of 4:5, find 'x'.

**Solution:**

$$\frac{3x+4}{4x+9} = \frac{16}{25}$$

$$\text{Or, } 75x+100 = 64x+144$$

$$\text{Or, } 75x-64x = 144-100$$

$$\text{Or, } 11x = 44$$

$$\text{Or } x = 4.$$

**Illustration 11.**

If  $\frac{2x-3}{3x-5}$  is the sub-duplicate ratio of 9:16; find 'x'.

**Solution:**

$$\frac{2x-3}{3x-5} = \frac{3}{4}$$

$$\text{Or, } 8x-12 = 9x-15$$

$$\text{Or, } -12+15 = 9x-8x$$

$$\text{Or, } 3 = x$$

**Illustration 12.**

Find  $x$  &  $y$  if 27,  $x$ , 12,  $y$  are in proportion.

**Solution:**

$$\frac{27}{x} = \frac{x}{12} = \frac{12}{y}$$

$$\text{Then } \frac{27}{x} = \frac{x}{12}$$

$$\text{Or, } x^2 = 27 \times 12$$

$$\text{Or, } x = 18.$$

$$\text{Now } \frac{x}{12} = \frac{12}{y}$$

$$\text{Or, } \frac{18}{12} = \frac{12}{y}$$

$$\text{Or } 18y = 144$$

$$\text{Or } y = 8.$$

**Illustration 13.**

If  $x, x+1, x+2, x+4$  are in proportion, find 'x'.

**Solution:**

$$\frac{x}{x+1} = \frac{x+2}{x+4}$$

$$\text{Or, } x^2+4x = x^2+2x+x+2$$

$$\text{Or, } 4x = 3x+2$$

$$\text{Or, } 4x-3x = 2$$

$$\text{Or, } x = 2$$

**Illustration 14.**

If  $x$  is mean proportional between  $x-2$  and  $x+6$ , find 'x'.

**Solution:**

$$\frac{x-2}{x} = \frac{x}{x+6}$$

$$\text{Or, } x^2 = x^2+6x-2x-12$$

$$\text{Or, } 4x = 12 \quad \text{or} \quad x = 3$$

**Illustration 15.**

If  $\frac{a}{2} = \frac{b}{3} = \frac{c}{5}$ , find value  $\frac{a+b+c}{a}$

**Solution:**

$$\frac{a}{2} = \frac{a+b+c}{2+3+5} = \frac{a+b+c}{10}$$

$$\text{Or, } \frac{10}{2} = \frac{a+b+c}{a} = 5.$$

**PRACTICE PROBLEMS**

- The ratio of the present age of a father to that of his son is 5:3. Ten years hence the ratio would be 3:2. Find their present ages. (Ans. 50, 30)
- The monthly salaries of two persons are in the ratio of 3:5. If each receives an increase of ₹20 in salary, the ratio is altered to 13:21. Find the respective salaries. (Ans. ₹240, ₹400)
- What must be subtracted from each of the numbers 17, 25, 31, 47 so that the remainders may be in proportion? (Ans. 3)
- If  $\frac{x}{b+c} = \frac{y}{c+a} = \frac{z}{a+b}$  show that  $(b-c)x + (c-a)y + (a-b)z = 0$
- If  $\frac{x}{y+z} = \frac{y}{z+x} = \frac{z}{x+y} = k$  prove that  $k = \frac{1}{2}$ , if  $x + y + z \neq 0$
- If  $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{1}{2}$  prove that  $\frac{a^2+ab+b^2}{a^2-ab+b^2} = \frac{91}{73}$

7. If  $\frac{a}{4} = \frac{b}{5} = \frac{c}{9}$  prove that  $\frac{a+b+c}{9} = 2$
8. If  $\frac{b+c}{a} = \frac{c+a}{b} = \frac{a+b}{c}$  and  $a+b+c \neq 0$  then show that each of these ratios is equal to 2. Also prove that  $a^2 + b^2 + c^2 = ab + bc + ca$
9. If  $a:b = c:d$  Show that  $xa + yb : a\alpha - b\beta = xc + yd : c\alpha - d\beta$
10. If  $\frac{\alpha}{q-r} = \frac{\beta}{r-p} = \frac{\gamma}{p-q}$  Prove that  $\alpha + \beta + \gamma = 0, p\alpha + q\beta + r\gamma = 0$
11. Monthly incomes of two persons Ram and Rahim are in the ratio 5:7 and their monthly expenditures are in the ratio 7:11. If each of them saves ₹60 per month. Find their monthly income. (Ans. ₹200, ₹280)
12. There has been increment in the wages of labourers in a factory in the ratio of 22:25, but there has also been a reduction in the number of labourers in the ratio of 15:11. Find out in what ratio the total wage bill of the factory would be increased or decreased. (Ans. 6:5 decrease)

### Variations:

#### Direct Variation:

If two variable quantities  $A$  and  $B$  be so related that as  $A$  changes  $B$  also changes in the same ratio, then  $A$  is said to vary directly as (or simply vary) as  $B$ . This is symbolically denoted as  $A \propto B$  (read as  $A$  varies as  $B$ )

#### Example:

The circumference of a circle  $= 2\pi r$ , so circumference of a circle varies directly as the radius, for if the radius increase (or decrease), circumference also increases or decreases.

From the above definition, it follows that:

If  $A$  varies as  $B$ , then  $A = KB$ , where  $K$  is constant ( $\neq 0$ )

Cor:  $A \propto B$  then  $B \propto A$  if  $A \propto B$ , then  $A = k B$ . or  $B = A/k$  i.e.,  $B \propto A$ .

**Example:** The area of a circle is directly proportional to radius of circle (i.e. if radius increases the area of a circle increases in the same ratio as radius) i.e.

$$A(\text{Area}) \propto r(\text{radius})$$

$$A = Kr$$

$$\text{Or, } K = A/r$$

$K$  is nothing but constant of proportionality

#### Inverse variation:

When two variable quantities  $A$  and  $B$  are such that the increase or decrease of  $A$  causes decrease or increase of  $B$  in a constant ratio then.  $A$  is said to be varies inversely as  $B$ . or  $A$  varies directly as the reciprocal of  $B$ .

i.e.  $A \propto 1/B$

Or,  $A = K/B$  [ $K = \text{Constant of proportionality}$ ]

Or,  $K = AB$

**Example:** Speed (S) is inversely proportional to time (t) i.e.

$$S \propto 1/t$$

Or,  $S = K/t$

Or,  $K = St$

### Joint variation:

A is said to vary jointly as B, C, and D.... if A varies directly as the product of (B, C, D....) i.e.

$$A \propto (BCD.....)$$

Or,  $A = K(BCD.....)$

**Example:** The volume of cuboid varies directly as the product of length (l), breadth (b), height (h)

i.e.  $V \propto lbh$

Or,  $V = Klbh$

Or,  $K = V/lbh$

### Some Elementary Results:

- (i) If  $A \propto B$ , then  $B \propto A$
- (ii) If  $A \propto B$  and  $B \propto C$ , then  $A \propto C$
- (iii) If  $A \propto B$  and  $B \propto C$ , then  $A - B \propto C$
- (iv) If  $A \propto C$  and  $B \propto C$ , then  $A - B \propto C$
- (v) If  $A \propto C$  and  $B \propto C$ , then  $\sqrt{AB} \propto C$
- (vi) If  $A \propto B$ , then  $A^n \propto B^n$ .
- (vii) If  $A \propto B$  and  $C \propto D$ , then  $AC \propto BD$  and  $A/C \propto B/D$
- (viii) If  $A \propto BC$ , then  $B \propto A/C$  and  $C \propto A/B$

### Illustration 16.

If  $a + b \propto a - b$  prove that

- (i)  $a \propto b$
- (ii)  $a^2 + b^2 \propto a^2 - b^2$
- (iii)  $a^2 + b^2 \propto ab$

**Solution:**

(i) Since  $a + b \propto a - b$ , then  $a + b = k(a - b)$ ,  $k$  is constant of variation. Or,  $(k - 1)a = (k + 1)b$

$$\text{Or, } \frac{a}{b} = \frac{k + 1}{k - 1}$$

Or,  $\frac{a}{b} = \frac{k + 1}{k - 1} \times b = mb$ , where  $m = \frac{k + 1}{k - 1}$ , a constant, Hence  $a \propto b$

(ii)  $\frac{a^2 + b^2}{a^2 - b^2} = \frac{m^2 b^2 + b^2}{m^2 b^2 - b^2} = \frac{m^2 + 1}{m^2 - 1} = a \text{ constant. So } a^2 + b^2 \propto a^2 - b^2$

(iii)  $\frac{a^2 + b^2}{ab} = \frac{m^2 b^2 + b^2}{mb^2} = \frac{(m^2 + 1)}{m^2} = a \text{ constant. So } a^2 + b^2 \propto ab$

**Illustration 17.**

If  $x + y \propto x - y$ , prove that  $ax + by \propto px + qy$ , where  $a, b, p, q$  are constants

**Solution:**

As  $x + y \propto x - y$ , So  $\frac{x + y}{x - y} = k$  (constant)

Or,  $x + y = k(x - y)$  Or,  $(k - 1)x = (k + 1)y$  Or,  $\frac{x}{y} = \frac{k + 1}{k - 1} = m$  (a constant), Or,  $x = my$

$\frac{ax + by}{px + qy} = \frac{amy + by}{pmy + qy} = \frac{am + b}{pm + q} = \text{constant. So } ax + by \propto px + qy$

**Illustration 18.**

If  $(a + b) \propto (a - b)$  show that  $(a^2 + b^2) \propto ab$

**Solution:**

As  $(a + b) \propto (a - b)$ , So  $(a + b) = k(a - b)$  Or  $a(k - 1) = b(k + 1)$ , Or,  $\frac{a}{b} = \frac{k + 1}{k - 1} = m$  (a constant)

$\frac{a^2 + b^2}{ab} = \frac{a}{b} + \frac{b}{a} = \frac{k + 1}{k - 1} + \frac{k - 1}{k + 1} = \text{Constant. So } (a^2 + b^2) \propto ab$

**Illustration 13.**

If the cost price of 12 kg. of rice is ₹10, what will be the cost of 15 kg. of rice?

**Solution:**

Let Cost and Quantity be denoted by  $A$  and  $B$ .

Given:  $A = ₹10$ ,  $B = 12$  kg. Now  $A \propto B$  i.e.,  $A = KB$  or,  $10 = K \cdot 12$  or,  $K = 10/12$

Now, we are to find  $A$ , when  $B = 15$  kg.

Again from  $A = KB$ , we have  $A = 10/12 \times 15 = ₹12.50$

**Illustration 19.**

A man can finish a piece of work, working 8 hours a day in 5 days. If he works now 10 hours daily, in how many days can he finish the same work?

**Solution:**

Let  $A$  = No. of days required to finish the work and  $B$  = Working hours per day

Given:  $A = 5, B = 8$ , it is clear that  $A \propto 1/B$

i.e.  $A = k \cdot 1/B$  or  $5 = k \cdot 1/8$  or  $k = 40$

To find  $A$  when  $B = 10$ , we have  $A = 40 \times 1/10 = 4$  days

**Illustration 20.**

One part of the total expenditure in a hostel is constant and the other part directly varies with the number of boarders. Per head expenditure is ₹100 when the number of boarders is 35 and ₹80 when the number of boarders is 45. If the number of boarders is 150 find the total expenditure of the boarders.

**Solution:**

Let  $n$  be the number of boarders and  $E$  be the total expenditure of the hostel. Part of the total expenditure is constant =  $F$  (let) and the other part of the total expenditure is  $V$  (let)

By question

$$V \propto n, \text{ then } V = Kn, \text{ where } K \text{ is constant}$$

$$\text{So, } E = F + V = F + Kn \dots \dots \dots (1)$$

$$\text{When } n = 35, \text{ then } E = 35 \times 100 = ₹ 3500$$

$$\text{When } n = 45, \text{ then } E = 45 \times 80 = ₹ 3600$$

$$\text{From (1) } 3500 = F + K35 \dots \dots \dots (2)$$

$$\text{From (2) } 3600 = F + k45 \dots \dots \dots (3)$$

$$\text{By (3) - (2) we have } 100 = 10K. \text{ So } K = 10$$

$$\text{In Equation (2) by putting } K = 10, \text{ we have } 3500 = F + 35 \times 10, \text{ So } F = ₹ 3150$$

$$\text{So from equation (1) we have } E = F + V = F + Kn = 3150 + 10n$$

$$\text{So if there are 150 boarders then total expenditure will be } E = 3150 + 10 \times 150 = ₹ 4650$$

**Illustration 21.**

If  $(5a - 2b) : (2a + b) = (6a - b) : (8a - b)$  find  $a : b$

**Solution:**

$$\text{Given } (5a - 2b) : (2a + b) = (6a - b) : (8a - b)$$

$$\text{Or, } \frac{5a - 2b}{2a + b} = \frac{6a - b}{8a - b}$$

$$\text{Or, } (5a - 2b)(8a - b) = (6a - b)(2a + b)$$

$$\text{Or, } 40a^2 - 5ab - 16ab + 2b^2 = 12a^2 - 2ab + 6ab - b^2$$

$$\text{Or, } 40a^2 - 21ab + 2b^2 = 12a^2 + 4ab - b^2$$

$$\text{Or, } 28a^2 - 25ab + 3b^2 = 0$$

$$\text{Or, } (4a - 3b)(7a - b) = 0$$

$$\text{So, } 4a - 3b = 0 \text{ and } 7a - b = 0$$

$$\text{Therefore } 4a = 3b \text{ i.e. } \frac{a}{b} = \frac{3}{4}$$

$$\text{and } 7a = b \text{ i.e. } \frac{a}{b} = \frac{1}{7}$$

# Time Value of Money and Annuity – Simple and Compound Interest

## 1.2

### Simple Interest

#### Interest:

Interest is the additional money which is paid by the borrower to the lender on the principal borrowed. The additional money (or) interest is paid for the use of money by the borrower. Interest is usually denoted by  $I$ .

For example:

Y borrowed ₹500 from Z for a year and returned ₹550. Here ₹50 is paid additionally. This ₹50 is the interest.

#### Rate of interest per annum:

Rate of interest per annum is the interest paid yearly for every ₹100 It is denoted by  $\frac{R}{100}$  or  $\frac{r}{100}$ .

#### Amount:

The sum of principal and interest paid is called as an Amount. It is denoted by  $A$ . Thus  $A = P + I$

#### Simple Interest:

If the interest is calculated uniformly on the original principal throughout the loan period it is called Simple Interest. It is denoted by S.I.

#### Formula:

Simple interest on the principal ' $P$ ' borrowed at the rate of  $r$  % p.a for a period of " $t$ " years is usually given by  $S.I = P \times r \times t / 100$

For example:

Gopi borrowed ₹1200 from Siva Reddy at 9% simple interest p.a for 3 years. How much Siva should get from Gopi as S.I.?

Sol:  $P = 1200, r = 9\%, t = 3$

$S.I = P \times r \times t / 100 = 1200 \times (9/100) \times 3 = ₹324$

#### Important Relations to remember:

1.  $S.I = P \times r \times t / 100$
2.  $A = P + S.I$

3.  $r = S.I \times 100/Pt$
4.  $t = S.I \times 100/Pr$
5.  $P = S.I \times 100/rt$
6.  $P = A - S.I$
7.  $S.I = A - P$
8.  $A = P \left( 1 + \frac{rt}{100} \right)$

**Illustration 22.**

Amit deposited ₹1,200 to a bank at 9% simple interest p.a. find the total interest that he will get at the end of 3 years.

**Solution:**

Here  $P = 1200$ ,  $r = 9/100 = 0.09$ ,  $t = 3$ ,  $I = ?$

$$I = P \times r \times t = 1200 \times 0.09 \times 3 = 324.$$

Amit will get ₹324 as interest.

**Illustration 23.**

Sumit borrowed ₹7,500 at 14.5% p.a S.I. for 2-1/2 year. Find the amount he had to pay after that period

**Solution:**

$P = 7500$ ,  $r = 14.5/100 = 0.145$ ,  $t = 2-1/2 = 2.5$ ,  $A = ?$

$$A = P \left( 1 + \frac{rt}{100} \right) = 7500 (1 + 0.145 \times 2.5) = 7500 (1 + 0.3625) = 7500 \times 1.3625 = 10218.75$$

Reqd. amount = ₹10,218.75

**Illustration 24.**

What sum of money will yield ₹1407 as interest in 1-1/2 year at 14% p.a simple interest?

**Solution:**

Here  $S.I = 1407$ ,  $t = 1.5$ ,  $r = 0.14$ ,  $P = ?$

$$S.I = P \times r \times t \text{ or, } 1407 = P \times 0.14 \times 1.5$$

$$\text{Or, } P = 1407 / (0.14 \times 1.5) = 6700$$

Required amount = ₹6,700

**Illustration 25.**

A sum of ₹1,200 was lent out for 2 years at S.I. The lender got ₹1,536 in all. Find the rate of interest.

**Solution:**

$$P = 1200, A = 1536, t = 2, r = ?$$

$$A = P(1 + rt/100) \text{ or, } 1536 = 1200(1 + r \times 2/100) = 1200 + 24r$$

$$\text{Or, } 1536 - 1200 = 24r \text{ or, } r = 336/24 = 14$$

Required rate = 14%

**Illustration 26.**

At what rate percent of Simple Interest will a sum, become double of itself in 5-1/2 years?

**Solution:**

$$A = 2P, P = \text{principal}, t = 5.5, r = ?$$

$$A = P(1 + rt/100) \text{ or, } 2P = P + P \times r \times t/100$$

$$1 = r \times 5.5 / 100$$

Required. Rate =  $r = 18.18\%$

**Illustration 27.**

In a certain time ₹1200 becomes ₹1560 at 10% p.a simple interest. Find the principal that will become ₹2,232 at 8% p.a in the same time.

**Solution:**

$$\text{In 1st case: } P = 1200, A = 1560, r = 0.10, t = ?$$

$$1560 = 1200 \{1 + t \times (.10)\} = 1200 + 120t$$

$$\text{Or, } 120t = 360 \text{ or, } t = 3$$

$$\text{In 2nd case: } A = 2232, t = 3, r = 0.08, P = ?$$

$$2232 = P \{1 + 3 \times (0.08)\} = P(1 + 0.24)$$

$$\text{Or, } P = 2232/1.24 = 1800$$

Required Principal is ₹ 1,800

**Illustration 28.**

Divide ₹2,760 in two parts such that simple interest on one part at 12.5% p.a for 2 years is equal to the simple interest on the other part at 12.5% p.a for 3 years.

**Solution:**

Let Investment in 1st case = X (say)

Investment in 2nd case = ₹ (2,760 – X)

Interest in 1st case =  $X \times 2 \times 0.125$

Interest in 2<sup>nd</sup> case =  $(2760 - X) \times 3 \times 0.125$

By question  $(2760 - X) \times 3 \times 0.125 = X \times 2 \times 0.125$

Or,  $2760 \times 3 \times 0.125 - 3 \times 0.125 \times X = 2 \times 0.125 \times X$

Or,  $1035 = 0.625X$

$X = 1035/0.625 = 1656$

So first part is ₹1,656 & second part is  $(2760 - 1656) = ₹1,104$

### Illustration 29.

A person borrowed ₹8,000 at a certain rate of simple interest for 2 years and then ₹10,000 at 1% lower than the first for 1 year. In all he paid ₹2,500 as interest in 3 years. Find the two rates at which he borrowed the amount.

#### Solution:

Let the rate of interest in the 1<sup>st</sup> case =  $r$ , so that in the 2<sup>nd</sup> case, rate of interest will be  $(r-1)$ . Now

$$8000 \times r \times 2/100 + 10000 \times (r-1) \times 1/100 = 2500$$

$$\text{Or, } 160r + 100r - 100 = 2500$$

$$\text{Or, } 260r = 2600$$

$$\text{Or, } r = 2600/260 = 10\%$$

So in first case interest is 10% and in second case it is  $(10-1) = 9\%$

### Illustration 30.

If the simple interest on ₹ 20,000 increases by ₹ 4,000 with the increase of time by 4 years. Find 'r'.

#### Solution:

$$r = \frac{\text{Increase S.I.}}{\text{Increase in time}} \times \frac{100}{P}$$

$$= \frac{4,000 \times 100}{20,000 \times 4} = 5\%$$

### Illustration 31.

If the difference between simple interest on ₹ 4,000 and on ₹ 6,500 for 5 yrs be 800 at the same rate of simple interest per annum. Find 'r'.

#### Solution:

$$S_1 - S_2 = \frac{(P_1 - P_2)rt}{100}$$

$$\text{Or, } 800 = \frac{(6500 - 4000)r \times 5}{100}$$

$$\text{Or, } r = 6.4\%$$

### Illustration 32.

If the simple interest on ₹ 300 increases by ₹ 75, when the rate of interest increases by 5% per annum. Find time.

**Solution:**

$$S_1 - S_2 = \frac{P(r_1 - r_2)t}{100}$$

Or,  $75 = \frac{300 \times 5 \times t}{100}$

Or,  $t = 5\text{yrs.}$

### Illustration 33.

Certain sum of money amounted to ₹ 2,600 at the end of 2nd yr. and amounted to ₹ 2,900 at the end of 3rd year at a certain rate of simple interest. Find sum.

**Solution:**

$$\text{S.I per year} = ₹ 2,900 - ₹ 2,600 = ₹ 300.$$

$$\text{S.I for 2yrs} = ₹ 600$$

$$\text{Sum} = ₹ 2,600 - ₹ 600 = ₹ 2,000.$$

### Illustration 34.

Ram lends ₹ 2000 at 4% S.I.P.A to Krishna and ₹ 3000 at 14% S.I.P.A to Hari. Find 'r' on the whole sum.

**Solution:**

$$r = \frac{2000 \times 4\% + 3000 \times 14\%}{2000 + 3000} = 10\%$$

**Calculation of interest on deposits in a bank:** Bank allows interest at a fixed rate on deposits from a fixed day of each month up to the last day of the month. Again interest may also be calculated by days.

## Time Value of Money

A rupee today will be more valuable than the same in future. Because a receipt today generate more preferable current consumption whereas a receipt tomorrow causes less preferable future consumption. To entice an individual for future receipt something extra is to be given with the present receipt.

This something extra is the compensation for foregoing current consumption – in financial term called **INTEREST**.

The interest can be used to express the time value of the money.

The time value of money analysis involves:

- What is ₹1 worth n years from today  $\Rightarrow$  Future value of a single flow?
- What is ₹1 to be deposited per year for n years' worth at the end of nth year from today  $\Rightarrow$  Future value of an annuity
- What is ₹1 to be received in nth year from today worth today  $\Rightarrow$  Present value of a single flow
- What is ₹1 to be received per year for n years' worth today  $\Rightarrow$  Present value of an annuity

## Applications

Time value of money analysis is applied

- (i) For pricing of Bonds.
- (ii) For pricing of stocks/firms.
- (iii) For finding out the value a new project contributes to the organization.
- (iv) For finding out the value of a particular division within an organization.
- (v) For loan amortization.
- (vi) For comparative study between rebate and low financing.

## Types of Valuation

### Future value of a single flow:

If an individual deposits ₹100 into a savings account at an interest rate of 10%, compounded annually then the deposit would grow as follows:

At the end of one year: Principal at the beginning of the period  $\Rightarrow$  ₹100

Interest for the one year period @ 10%  $\Rightarrow$  ₹10

-----

Value of the original amount at the end of the year  $\Rightarrow$  ₹110

At the end of 2nd year: Principal at the beginning of the period  $\Rightarrow$  ₹110

Interest for the one year period @ 10%  $\Rightarrow$  ₹ 11

-----

Value of the original amount at the end of the 2nd year  $\Rightarrow$  ₹121

As shown above the interest amounts to ₹11 in the second year composed of ₹10 as interest on the original principal of ₹100 in the first year and  $(10/100) \times 10$  i.e. ₹1 as *interest on the interest* of the first year in the second year. *This interest on interest principle is called interest compounded annually.*

So if 'P' is deposited for 'n' years at an interest rate 'r%', compounded annually, then the amount P would be

- At the end of 1<sup>st</sup> year  
 $= FV_1 = P \times (1+r)^1$ , where  $FV_1$  is the future value of P at end of year 1
- At the end of 2<sup>nd</sup> year  
 $= FV_2 = P \times (1+r)^2$  where  $FV_2$  is the future value of P at end of year 2
- At the end of 3<sup>rd</sup> year  
 $= FV_3 = P \times (1+r)^3$  where  $FV_3$  is the future value of P at end of year 3  
 .....
- At the end of n<sup>th</sup> year  
 $= FV_n = P \times (1+r)^n$

where  $FV_n$  is the future value of  $P$  at end of year .....(1)

In general equation (1) gives  $FV_n$  — the future value of a deposit at the end of  $n$ th period if  $r$  is the interest rate per annum.

Equation (1) is the basic equation in compounding analysis.

The factor  $(1 + r)^n$  is termed as

- Compounding factor or
- The Future Value Interest Factor (FVIF) or
- Terminal value Interest Factor.

So equation (1) can be alternatively written as

$FV_n = P \times FVIF_{r,n}$ , where  $FVIF_{r,n}$  is the future value interest factor at  $r\%$  for  $n$  periods

Tables have been constructed for  $FVIF_{r,n}$  for wide ranges of  $r$  &  $n$ . Such table is given at the end of all finance book.

So if the interest is added after certain period of time (say yearly, half yearly, quarterly and monthly etc) to the principal so that the amount at the end of the period becomes the principal for the next time period then the total interest paid over all the time period is called the Compound Interest. It is usually denoted by C.I.

Compound Interest = Amount – Principal =  $P \times FVIF_{r,n} - P = P \times (1 + r)^n - P$

### Illustration 25.

If Mr. X deposits ₹5,000 today in a bank which pays 8% interest compounded annually, how much the deposit will grow at the end of 10<sup>th</sup> year ?

#### Solution:

Here we are going to find  $FV_n = P \times FVIF_{r,n}$ , where  $P = ₹5,000$ ,  $n = 10$  and  $FV_{8\%,10} = 2.159$ . So  $FV_{10} = 5000 \times 2.159 = 10795$ .

### Future value of an Annuity

Annuity is a periodical flow of money, where at equal interval of time equal quantum of money flows. So a flow of money will be considered as annuity if two factors are satisfied-

- Interval between any two successive flows is same;
- Quantum of money flow in each interval is same;

These two factors are to be satisfied simultaneously. Breaking of any one of these or both, will constitute a multiple flow but not an annuity.

e.g.

- ₹5,000 is deposited at the end of every month is an annuity – interval between successive deposit is unchanged and amount of deposit is also remained same.
- ₹10,000 is deposited at the beginning of every quarter is an annuity – interval between successive deposit is unchanged and amount of deposit is also remained same.

- (c) ₹5,000 is deposited at the end of every month from January to March, ₹8,000 is deposited at the end of every month from April to September and ₹10,000 is deposited at the end of every month from October to December. It is a multiple flow but not an annuity – although interval between successive deposits is unchanged but there are changes in amount of deposit.
- (d) ₹10,000 is deposited at the end of first year, ₹10,000 is deposited at the end of 4<sup>th</sup> year and ₹10,000 is deposited at the end of 9<sup>th</sup> year. It is a multiple flow but not an annuity – although there is no change in the amount of deposit but interval between successive deposits is changed.
- (e) ₹10,000 is deposited at the end of first quarter, ₹20,000 is deposited at the end of 3<sup>rd</sup> quarter and ₹18,000 is deposited at the end of 4<sup>th</sup> quarter. It is a multiple flow but not an annuity – because here interval between successive flows as well as quantum, both changed. No deposit is made at the end of 2<sup>nd</sup> quarter and every time deposit amount varied.

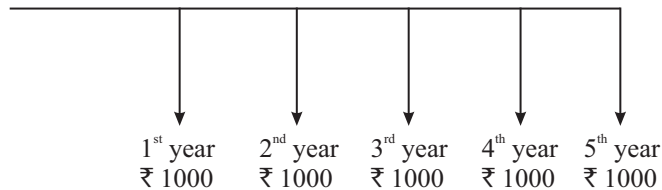
Annuities are of two types:

- (i) Regular annuity or deferred annuity when flow of money occurs at the end of a period;
- (ii) Annuity due when flow of money occurs at the beginning of a period;

Suppose an individual will deposit ₹1,000 in a bank at the end of each year for next five years. Also suppose bank gives 10% interest p.a. What will be the matured amount at the end of the fifth year?

Here there will be an annuity because interval between successive deposits is same (one year) and quantum of deposit in each case is also same (₹1000). Moreover since deposit is made at the end of a period, therefore the annuity is Regular annuity.

Total matured amount comprises of –



1. Future value of ₹1,000, deposited at the end of 1<sup>st</sup> year matured at the end of 5<sup>th</sup> year  $FV_n = P \times FVIF(r, n)$  i.e.  $FV_4 = 1000 \times FVIF(10\%, 4) = 1000 \times 1.464$  (from table) = ₹ 1464;
2. Future value of ₹1,000, deposited at the end of 2<sup>nd</sup> year matured at the end of 5<sup>th</sup> year  $FV_n = P \times FVIF(r, n)$  i.e.  $FV_3 = 1000 \times FVIF(10\%, 3) = 1000 \times 1.331$  (from table) = ₹ 1331;
3. Future value of ₹1,000, deposited at the end of 3<sup>rd</sup> year matured at the end of 5<sup>th</sup> year  $FV_n = P \times FVIF(r, n)$  i.e.  $FV_2 = 1000 \times FVIF(10\%, 2) = 1000 \times 1.210$  (from table) = ₹ 1210;
4. Future value of ₹1,000, deposited at the end of 4<sup>th</sup> year matured at the end of 5<sup>th</sup> year  $FV_n = P \times FVIF(r, n)$  i.e.  $FV_1 = 1000 \times FVIF(10\%, 1) = 1000 \times 1.100$  (from table) = ₹ 1100;
5. Future value of ₹1,000, deposited at the end of 5<sup>th</sup> year matured at the end of 5<sup>th</sup> year  $FV_n = P \times FVIF(r, n)$  i.e.  $FV_0 = 1000 \times FVIF(10\%, 0) = 1000 \times 1.000$  (from table) = ₹ 1000;

Sum total of (1) to (5) will be the matured amount of the annuity at the end of 5<sup>th</sup> year i.e. ₹6105

But instead of computing the matured amount of a regular annuity in the aforesaid manner, general formula is made and that is

$$FV_n = P \times \frac{(1+r)^n - 1}{r} \dots \dots \dots (2)$$

Where,

$FV_n$  = Future value;

$P$  = Amount deposited at the end of each period;

$r$  = Rate of interest per period;

$n$  = Duration of the annuity;

$\frac{(1+r)^n - 1}{r}$ : This factor is called *Future Value Interest Factor for Annuity (Regular)* for period  $n$  at an interest rate of  $r$  p.a or  $FVIF_{r,n}$

Tables have been constructed for  $FVIF(r, n)$  for wide ranges of  $r$  &  $n$ .

So in our illustrative problem  $P = ₹1000, r = 10\%$  p.a,  $n = 5$  yrs;

Therefore  $FV_n = P \times \frac{(1+r)^n - 1}{r} = P \times FVIF_{10\%,5} = 1000 \times 6.105$  (from table) = ₹6105, same as in our previous calculation.

In case of Annuity due equation (2) will be slightly modified as follows:

$$FV_n = P \times \frac{[(1+r)^n - 1] \times (1+r)}{r} \dots \dots \dots (2A)$$

**Present value of a single flow**

Suppose an individual will receive ₹1,000 from a bank deposit three years hence for his deposit today at an interest rate of 10 per cent, compounded annually.

From our knowledge on FUTURE VALUE of a single flow we can safely conclude that ₹1000 in our example is the future value of a single flow i.e. the deposit amount of today. If this deposit amount we consider as  $P$  then we can write:

$$1000 = P \times (1 + 0.10)^3$$

$$\text{Or, } 1000 = P \times FVIF (10\%, 3)$$

$$\text{Or, } P = 1000 / (1+.10)^3 \text{ OR } 1000 / FVIF (10\%, 3)$$

The ‘ $P$ ’ in the above case is present day deposit .In business term it is called *PRESENT VALUE OF A SINGLE FLOW* occurring in future. The factor  $1 / (1+.10)^3$  or  $1 / FVIF (10\%,3)$  or reciprocal of future value interest factor is called *PRESENT VALUE INTEREST FACTOR* or *the DISCOUNTING FACTOR*.

Present value calculations of future cash flows allow us to place all the future cash flows on a current footing so that comparisons can be made in terms of today’s flows.

So general formula for present value of a single flow can be obtained by manipulating the future value formula and it is:

$$PV = \frac{FV}{(1+r)^n} \dots \dots \dots (3)$$

The factor  $\frac{1}{(1+r)^n}$  is called *Present value interest factor of a single flow or PVIF* at  $r$  percent for  $n$  years.

Interest rate in the above formula is also called discount rate or the capitalization rate.

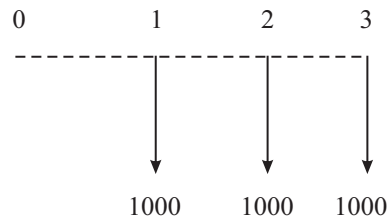
A present value table containing PVIFs for wide ranges of ‘ $r$ ’ and ‘ $n$ ’ is available.

Present value bears the following characteristics in contrast to future value:

- Greater the interest rate, lower is the present value.
- Interest compounding period more, lower is the present value.

### Present value of an annuity

Suppose an individual is expecting to receive ₹1000 annually for 3 years starting from 1 year hence. What is the value of such annuity receipts right now i.e. today (time period 0). The following diagrammatic representation of the above case will be helpful for better understanding:



From the above it is clear that finding the present value of an annuity is nothing but finding the sum of a series of individual present values. Therefore we can write the general formula for the present value of an ordinary annuity of magnitude ‘ $A$ ’ for  $n$  period at a discounting rate (rate of interest) ‘ $r$ ’ as:

$$PVA = \frac{A}{(1+r)^1} + \frac{A}{(1+r)^2} + \frac{A}{(1+r)^3} + \dots \dots \dots + \frac{A}{(1+r)^n}$$

$$PVA = A \frac{(1+r)^n - 1}{r(1+r)^n} \dots \dots \dots (4)$$

The factor  $\frac{(1+r)^n - 1}{r(1+r)^n}$  is called *Present value interest factor of an (ordinary) annuity* or PVIFA in short.

A present value table containing PVIFAs for wide ranges of ‘ $r$ ’ and ‘ $n$ ’ is available

$$PVIFA = \frac{(1+r)^n - 1}{r(1+r)^n}$$

Or,  $PVIFA = \frac{(1+r)^n - 1}{r} \times \frac{1}{(1+r)^n}$

Or,  $PVIFA = FVIFA_{r,n} \times PVIF_{r,n}$

**Illustration 35.**

Mr. X will receive ₹1500.00 at the end of every year starting from today and this will continue for next 5 years. How much deposit he has to make today if the interest rate is 10% p.a.?

**Solution:**

This is clearly a problem of present value. Since the receipts in the hands of Mr. X are of same magnitude and also occurring in equal intervals of time – at the end of every year. So receipts form regular annuity. Present value of such annuities as per formula 4 is

$$PVA = A \frac{(1+r)^n - 1}{r(1+r)^n}$$

Where  $A = 1500$ ,  $r = 10\% = 0.10$  &  $n = 5$

$$\text{Or, } PVA = 1500 \times \frac{(1+0.1)^5 - 1}{0.1 \times (1+0.1)^5} = ₹ 5686.18$$

OR

$$PVA = A \times FVIFA_{r,n} \times PVIF_{r,n}$$

$$= 1500 \times FVIFA_{10\%,5} \times PVIF_{10\%,5}$$

$$= 1500 \times 6.105 \times 0.621 = 5686.801$$

Formula 4 is applicable in case of regular annuity.

In case of Annuity Due the formula for present value is given by

$$PVA = A \frac{(1+r)^n - 1}{r(1+r)^n} \times (1+r) \dots \dots \dots (4A)$$

So from formula (1), (2), (2A), (3), (4) & (4A) we can summarize the following formula for unit amount (i.e.  $A = 1$ ), for interest rate ' $r$  p.a.' and for ' $n$  years' for ready reference and comparative study.

FLOW	FUTURE VALUE	PRESENT VALUE
Single	$(1+r)^n$	$\frac{1}{(1+r)^n}$
Regular Annuity	$\frac{(1+r)^n - 1}{r}$	$\frac{(1+r)^n - 1}{r(1+r)^n}$
Annuity Due	$\frac{[(1+r)^n - 1] \times (1+r)}{r}$	$\frac{(1+r)^n - 1}{r(1+r)^n} \times (1+r)$

In all the above cases we have considered Compounded Annually. But if compounding i.e. interest on interest is given more than once in a year then formulae need modification. More than once compounding will be treated in next note.

### Shorter compounding Period

Till now we have our all computations assuming interest rate ‘*r*’ on per annum basis i.e. all flows are compounded annually. If compounding is done for shorter period then different results will arrive. Suppose Mr y deposited ₹1000 in a company deposit which gives 5% interest on the balance amount at the end of every six months i.e. compounding is done semi-annually. So

Principal at the beginning	= ₹1000.00
Interest for the first 6 month	= ₹1000 × 5/100 × 6/12 = ₹25.00
<hr style="border: 0.5px solid black;"/>	
Total amount at the end of 6 month	= ₹1025.00

This ₹1025 will be the principal amount at the beginning of the next 6 months. Result will be

Principal at the beginning	= ₹1025.00
Interest for the first 6 month	= ₹1025 × 5/100 × 6/12 = ₹25.625
<hr style="border: 0.5px solid black;"/>	
Total amount at the end of next 6 month	= ₹1050.625

If compounding is done annually total amount at the end of one year will be ₹1050.00 instead of ₹1050.625. Extra amount is ₹0.625. This is nothing but the interest for the last six months on interest for the first six months i.e.

Interest for the first six months = ₹ 25.00

Interest on this amount @ 5% p.a. for the last six months =  $25 \times 5/100 \times 1/2 = ₹0.625$ . So in one year if compounding is done every six months the maturity value will be

$$\begin{aligned} ₹1050.625 &= 1000 \times (1.050625) \\ &= 1000 \times (1 + 0.05625) \\ &= 1000 \times \left(1 + \frac{5}{200}\right)^2 \\ &= 1000 \times \left\{1 + \frac{5}{2 \times 100}\right\}^{2 \times 1} \dots \dots \dots (7) \end{aligned}$$

In formula (7), first ‘2’ represents no of times compounding is done in one year for semi-annual compounding.

Therefore general formula for future value of a single flow ₹ P, deposited for ‘*n*’ years at an interest rate of ‘*r* % p.a.’ with compounding done ‘*m*’ times in a year is

$$FV = P \times \left(1 + \frac{r}{m}\right)^{m \times n} \dots \dots \dots (8)$$

Following the same principle the general formula for different types of flows for ‘*n*’ years at an interest rate of ‘*r* % p.a.’ with compounding done ‘*m*’ times in a year are given below for an unit amount (i.e. *A* = 1):

FLOW	FUTURE VALUE	PRESENT VALUE
Single	$(1 + r/m)^{mn}$	$\frac{1}{(1 + r/m)^{mn}}$
Regular Annuity	$\frac{(1 + r/m)^{mn} - 1}{r/m}$	$\frac{(1 + r/m)^{mn} - 1}{r/m(1 + r/m)^{mn}}$
Annuity Due	$\frac{[(1 + r/m)^{mn} - 1] \times (1 + r/m)}{r/m}$	$\frac{(1 + r/m)^{mn} - 1}{r/m(1 + r/m)^{mn}} \times (1 + r/m)$

### Illustration 36.

Mr. P opened a recurring deposit account in a bank, started depositing semi-annual instalments in arrear amounting ₹ 5000 for 5 years. If the rate of compound interest is 6% p.a. what will be the maturity amount.

#### Solution:

It is a problem of future value of regular annuity when compounding is done more than once in a year – in the given problem it is twice in a year. So the matured amount will be:

$$\text{Amount} = 5000 \times FVIFA_{\left(\frac{r}{m}, mn\right)} = 5000 \times FVIFA_{\left(\frac{6}{2}, 5 \times 2\right)} = 5000 \times FVIFA_{(3\%, 10)}$$

$$= 5000 \times 11.464 \text{ (from table)} = ₹57320$$

### Doubling Period

A frequently asked question in finance is “how long it would take to double the invested amount at a given interest rate?”

A rule of thumb called “Rule of 72” is available to answer this question. This rule states that the doubling period will be obtained by dividing 72 by the given interest rate.

For example if interest rate is 6 percent then doubling period will be 72/6 i.e. 12 years.

Another thumb rule of doubling period is named “Rule of 69”. As per this rule the doubling period will be = 0.35 + (69 / interest rate)

### Effective versus Nominal Interest Rate

While discussing the shorter compounding period it is found that ₹1000 if deposited for one year @5% p.a. compounded semi-annually, the maturity amount will be ₹1050.625. This means that ₹1000 grows at the rate of 5.0625% p.a. This 5.0625% is called the effective rate of interest.

So effective rate of interest is defined as that interest rate which under annual compounding produces the same result as that produced by an interest rate under shorter compounding period. The general relationship between effective interest rate and nominal interest rate is as follows:

$$\text{Effective rate} = \left(1 + \frac{r}{m}\right)^m - 1$$

$m$ , the no of times compounding is done in a period

**Illustration 37.**

Find the compound interest on ₹1000 for 4 years at 5% p.a

**Solution:**

Here  $P = ₹1000$ ,  $n = 4$ ,  $r = 0.05$ ,  $A = ?$

We have  $FV_n = A = P \times (1+r)^n = 1000 \times (1+0.05)^4 = 1000 \times 1.216 = 1216$

So Compound interest =  $A - P = 1216 - 1000 = ₹216$

**Illustration 38.**

In what time will a sum of money double itself at 5% p.a C.I.

**Solution:**

Here,  $P = P$ ,  $A = 2P$ ,  $r = 0.05$ ,  $n = ?$

We have  $FV_n = A = P \times (1+r)^n$

Or,  $2P = P \times (1+0.05)^n$

Or,  $2 = (1+0.05)^n$

Or,  $\log 2 = n \log 1.05$  Or,  $n = \frac{\log 2}{\log 1.05} = 14.2$  years

From rule of 72 approximate doubling period =  $\frac{72}{5} = 14.4$  years

**Illustration 39.**

The difference between simple and compound interest on a sum put out for 5 years at 3% was ₹46.80. Find the sum.

**Solution:**

Let the sum is  $X$

Simple Interest =  $X \times 0.03 \times 5$

Compound interest =  $X(1+0.03)^5 - X$

So by question we can write  $[X(1+0.03)^5 - X] - [X \times 0.03 \times 5] = 46.80$

Or,  $X[(1+0.03)^5 - 1 - 0.15] = 46.80$

Or,  $X[1.16 - 1 - 0.15] = 46.80$

Or,  $X[1.16 - 1.15] = 46.80$

Or,  $X[0.01] = 46.80$

Or,  $X = \frac{46.80}{0.01} = ₹ 4,680$  (approx.)

**Illustration 40.**

What is the present value of ₹1,000 due in 2 years at 5% compound interest p.a, according as the interest is paid (a) yearly (b) half-yearly?

**Solution:**

(a) We know present value  $P = \frac{FV}{(1+r)^n}$ , where interest is paid yearly

$$\text{Or, } P = \frac{1000}{(1+0.05)^2} = \frac{1000}{1.1025} = ₹ 907 \text{ (approx)}$$

(b) We know present value  $P = \frac{FV}{(1+r/m)^{mn}}$ , where interest is paid  $m$  times in a year

In the given case interest is paid semiannually i.e.  $m = 2$

$$P = \frac{FV}{(1+r/m)^{mn}} = \frac{1000}{\left(1 + \frac{0.05}{2}\right)^{2 \times 2}} = ₹ 906 \text{ (approx)}$$

**Illustration 41.**

A sum of money invested at C.I. payable yearly amounts to ₹10,816 at the end of the second year and to ₹11,248.64 at the end of the third year. Find the rate of interest and the sum.

**Solution:**

$$\text{We have } FV_n = A = P \times (1+r)^n$$

$$\text{In First case: } FV_n = A = 10816 = P \times (1+r)^2 \dots \dots \dots (1)$$

$$\text{In Second case: } FV_n = A = 11248.64 = P \times (1+r)^3 \dots \dots \dots (2)$$

$$\text{Dividing (2) by (1): } \frac{11248.64}{10816} = \frac{P \times (1+r)^3}{P \times (1+r)^2} = (1+r)$$

Or,  $1.04 = 1 + r$  Or,  $r = 0.04$  Or,  $r = 4\%$ . This is the required rate of interest.

From (1) we have  $A = P \times (1+r)^n$  Or,  $10816 = P(1.04)^2$ . So  $P = ₹10,000$ . This is the required sum

**Illustration 42.**

Calculate the value 5 years hence of a deposit of ₹1,000 made today when the interest rate is 8% p.a. if (i) simple interest is given (ii) compound interest is given

**Solution:**

(i) On simple interest

$$\text{Interest amount at the end of 5 years} = 1000 \times 0.08 \times 5 = ₹ 400$$

So the maturity amount will be =  $1000 + 400 = ₹1,400$

(ii) On compound interest

The maturity amount  $A = 1000 \times (1+0.08)^5 = 1000 \times 1.469 = ₹1,469$

#### Illustration 43.

Find the present value of ₹1,000 receivable 6 years hence if the rate of compound interest is 6% p.a.

**Solution:**

Let the present value =  $P$

So by question we can write  $1000 = P \times (1 + 0.06)^6 = P \times (1.06)^6$

$$\text{So, } P = \frac{1000}{(1.06)^6} = ₹704.96$$

#### Illustration 44.

A finance company offers to give ₹8000 after 12 years in return for ₹1000 deposited today. Find out the approx rate of interest offered p.a?

**Solution:**

By question we can write  $8000 = 1000 \times (1+r)^{12}$

We have to find out the value of  $r$

We can write

$$\frac{8000}{1000} = (1+r)^{12}$$

$$\text{Or, } 8 = (1+r)^{12}$$

$$\text{Or, } \text{Log } 8 = 12 \text{ Log}(1+r)$$

$$\text{Or, } 0.90309 = 12 \text{ Log}(1+r)$$

$$\text{Or, } 0.0752 = \text{Log}(1+r) \quad \text{Or, } 10^{0.0752} = 1+r$$

$$\text{Or, } 1.19 = 1+r, \quad \text{Or, } r = 1.19 - 1 = 0.19$$

$$\text{Or, } r = 19\%$$

#### Illustration 45.

A bank offers 8% nominal rate of interest with quarterly compounding. What is the effective rate of interest?

**Solution:**

We know that effective rate of interest  $r = \left(1 + \frac{k}{m}\right)^m - 1$

Where

$r$  = effective rate of interest

$k$  = nominal rate of interest

$m$  = no of times interest is given in a year

So by question we can write  $r = \left(1 + \frac{0.08}{4}\right)^4 - 1$

Or,  $r = (1.02)^4 - 1 \cong 0.082 = 8.2\%$

#### Illustration 46.

A man invests ₹100 at the end of 2001, ₹200 at the end of 2002, ₹300 at the end of 2003, ₹400 at the end of 2004 & ₹500 at the end of 2005. If all interest accumulates at 5% p.a. what do his investments amount to at the end of 2006?

#### Solution:

By question we can write

$$A_1 = 100 \times (1.05)^5 = 127.6282$$

$$A_2 = 200 \times (1.05)^4 = 243.1013$$

$$A_3 = 300 \times (1.05)^3 = 347.2875$$

$$A_4 = 400 \times (1.05)^2 = 441$$

$$A_5 = 500 \times (1.05)^1 = 525$$

So at the end of 2006 total investment amounts to  $A_1 + A_2 + A_3 + A_4 + A_5 = ₹1,684.02$

#### Illustration 47.

Mr S invests in a bank ₹X on the first day of 2006. In the subsequent years on the 1<sup>st</sup> of January he deposits money double that of the money deposited in the previous year after withdrawing the interest only on the same day. It was found that balance in his account on 2<sup>nd</sup> January 2015 was ₹2046. Find out the amount he deposited on 1<sup>st</sup> January 2006.

#### Solution:

Let the money invested on 1<sup>st</sup> January 2006 be X.

Then by question we can write

$X + 2X + 4X + \dots \dots \dots 10 \text{ terms} = 2046$  [As, time period between 1st January 2006 to 1st January 2015 is 10 years.]

$$\text{Or, } X(2^0 + 2^1 + \dots \dots \dots + 2^9) = 2046$$

$$\text{Or, } X \frac{(2^{10} - 1)}{2 - 1} = 2046$$

$$\text{Or, } 1023X = 2046$$

$$\text{Or, } X = ₹2$$

**Illustration 48.**

Find the interest on ₹1000 for 10 years at 4% p.a., the compound interest being paid quarterly

**Solution:**

By question we can write

$$A = 1000 \times \left(1 + \frac{0.04}{4}\right)^{4 \times 10}$$

$$\text{Or, } A = 1000 \times (1.01)^{40}$$

$$\text{Or, } A = 1488.864 = 1488.86 \text{ (approx.)}$$

$$\text{So Interest} = 1488.86 - 1000 = ₹488.86$$

**Illustration 49.**

Someone promises to give you ₹5,000 after 10 years in exchange for ₹1,000 today. What interest rate compounded annually is implicit in the offer?

**Solution:**

By question we can write

$$A = P \times (1+r)^{10}$$

$$\text{Or, } 5000 = 1000 \times (1+r)^{10}$$

$$\text{Or, } 5 = (1+r)^{10}$$

$$\text{Or, } \text{Log } 5 = 10 \text{ Log}(1+r)$$

$$\text{Or, } 0.69897 = 10 \text{ Log}(1+r)$$

$$\text{Or, } 0.0699 = \text{Log}(1+r) \quad \text{Or, } 1+r = 10^{0.069}$$

$$\text{Or, } 1.175 = 1+r$$

$$\text{Or, } r = 0.175 = 17.5\%$$

**Illustration 50.**

If ₹1 was deposited in the post office savings bank on 1<sup>st</sup> April, 2003, what would it amount to on 1<sup>st</sup> April 2033 if compound interest at 3% p.a. was allowed?

**Solution:**

By question we can write

$$A = 1 \times (1+0.03)^{30}$$

$$\text{Or, } A = (1.03)^{30} = ₹2.43 \text{ (approx.)}$$

**Illustration 51.**

A sum of money invested at compound interest amounts to ₹21632 at the end of 2<sup>nd</sup> year and ₹22497.28 at the end of third year. Find the rate of interest and sum invested.

**Solution:**

By question we can write

$$21632 = P \times (1+r)^2 \dots \dots \dots (1)$$

$$22497.28 = P \times (1+r)^3 \dots \dots \dots (2)$$

By (2) / (1) we get

$$\frac{22497.28}{21632} = \frac{(1+r)^3}{(1+r)^2}$$

$$\text{Or, } 1.04 = 1+r \quad \text{Or, } r = 0.04 = 4\%$$

Now we can write

$$21632 = P(1.04)^2 \quad \text{Or, } P = ₹20,000$$

**Illustration 52.**

What is the difference between the effective rate of interest and nominal rate of interest in the following case?  
“Nominal rate of interest is 24% and the frequency of compounding is twelve times a year”

**Solution:**

We know that effective rate of interest  $r = \left(1 + \frac{k}{m}\right)^m - 1$

Where

$r$  = effective rate of interest

$k$  = nominal rate of interest

$m$  = no of times interest is given in a year

$$\text{So by question we can write } r = \left(1 + \frac{0.24}{12}\right)^{12} - 1$$

$$\text{Or, } r = (1.02)^{12} - 1 \cong 0.268 = 26.8\%$$

This is the required effective rate of interest.

**Illustration 53.**

Mr. Raghu deposits ₹10,000 in a bank now. The interest rate is 10% and compounding is done semi-annually. What will the deposit grow to after 10 years?

**Solution:**

By question we can write  $A = P \times \left(1 + \frac{r}{m}\right)^{m \times t}$

Here  $P = 10000$ ,  $r = 0.10$ ,  $m =$  no. times of interest is given in year  $= 2$ ,  $t = 10$

$$\text{So we can write } A = 10000 \times \left(1 + \frac{0.1}{2}\right)^{2 \times 10}$$

$$\text{Or, } A = 10000 \times (1.05)^{20} \quad \text{Or, } A = ₹26532.98$$

**Illustration 54.** The annual birth and death rate per 1000 are 39.4 & 19.4. How many years population will be double.

**Solution:**

Population increase per year =  $39.4 - 19.4 = 20$  per 1000.

Population increase = 2%

$$2P = P(1+i)^n$$

Or,  $2 = (1+0.02)^n = (1.02)^n$

Or,  $\log 2 = n \log(1.02)$

Or,  $\frac{0.3010}{0.0086} = n = 35$  yrs.

**Illustration 55.**

In what rate of compound interest certain sum of money becomes 27 times of itself in 3 years.

**Solution:**

$$r = (27^{\frac{1}{3}} - 1) \times 100 = 200\%$$

**Illustration 56.**

Find the difference between C.I & S.I for the sum of ₹ 625 at 8% p.a for 2 yrs.

**Solution:**

$$\text{C.I.S.I} = 625 \left( \frac{8}{100} \right)^2 = ₹ 4.$$

**Table 1: Future Value Interest Factor (FVIF) (\$1 at r% for n periods)**

$FVIF = (1+r)^n$  ;  $FV = PV (FVIF_{r,n})$

n/r	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%	13%	14%	15%	16%	17%	18%	19%	20%	25%	30%	40%
0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	1.01	1.02	1.03	1.04	1.05	1.06	1.07	1.08	1.09	1.10	1.11	1.12	1.13	1.14	1.15	1.16	1.17	1.18	1.19	1.20	1.25	1.30	1.40
2	1.02	1.04	1.06	1.08	1.10	1.12	1.14	1.17	1.19	1.21	1.23	1.25	1.28	1.30	1.32	1.35	1.37	1.39	1.42	1.44	1.56	1.69	1.96
3	1.03	1.06	1.09	1.12	1.16	1.19	1.23	1.26	1.30	1.33	1.37	1.40	1.44	1.48	1.52	1.56	1.60	1.64	1.69	1.73	1.95	2.20	2.74
4	1.04	1.08	1.13	1.17	1.22	1.26	1.31	1.36	1.41	1.46	1.52	1.57	1.63	1.69	1.75	1.81	1.87	1.94	2.01	2.07	2.44	2.86	3.84
5	1.05	1.10	1.16	1.22	1.28	1.34	1.40	1.47	1.54	1.61	1.69	1.76	1.84	1.93	2.01	2.10	2.19	2.29	2.39	2.49	3.05	3.71	5.38
6	1.06	1.13	1.19	1.27	1.34	1.42	1.50	1.59	1.68	1.77	1.87	1.97	2.08	2.19	2.31	2.44	2.57	2.70	2.84	2.99	3.81	4.83	7.53
7	1.07	1.15	1.23	1.32	1.41	1.50	1.61	1.71	1.83	1.95	2.08	2.21	2.35	2.50	2.66	2.83	3.00	3.19	3.38	3.58	4.77	6.27	10.54
8	1.08	1.17	1.27	1.37	1.48	1.59	1.72	1.85	1.99	2.14	2.30	2.48	2.66	2.85	3.06	3.28	3.51	3.76	4.02	4.30	5.96	8.16	14.76
9	1.09	1.20	1.30	1.42	1.55	1.69	1.84	2.00	2.17	2.36	2.56	2.77	3.00	3.25	3.52	3.80	4.11	4.44	4.79	5.16	7.45	10.60	20.66
10	1.10	1.22	1.34	1.48	1.63	1.79	1.97	2.16	2.37	2.59	2.84	3.11	3.39	3.71	4.05	4.41	4.81	5.23	5.69	6.19	9.31	13.79	28.93
11	1.12	1.24	1.38	1.54	1.71	1.90	2.10	2.33	2.58	2.85	3.15	3.48	3.84	4.23	4.65	5.12	5.62	6.18	6.78	7.43	11.64	17.92	40.50
12	1.13	1.27	1.43	1.60	1.80	2.01	2.25	2.52	2.81	3.14	3.50	3.90	4.33	4.82	5.35	5.94	6.58	7.29	8.06	8.92	14.55	23.30	56.69
13	1.14	1.29	1.47	1.67	1.89	2.13	2.41	2.72	3.07	3.45	3.88	4.36	4.90	5.49	6.15	6.89	7.70	8.60	9.60	10.70	18.19	30.29	79.97
14	1.15	1.32	1.51	1.73	1.98	2.26	2.58	2.94	3.34	3.80	4.31	4.89	5.53	6.26	7.08	7.99	9.01	10.15	11.42	12.84	22.74	39.37	111.12
15	1.16	1.35	1.56	1.80	2.08	2.40	2.76	3.17	3.64	4.18	4.78	5.47	6.25	7.14	8.14	9.27	10.54	11.97	13.59	15.41	28.42	51.19	155.57
16	1.17	1.37	1.60	1.87	2.18	2.54	2.95	3.43	3.97	4.59	5.31	6.13	7.07	8.14	9.36	10.75	12.33	14.13	16.17	18.49	35.53	66.54	217.80
17	1.18	1.40	1.65	1.95	2.29	2.69	3.16	3.70	4.33	5.05	5.90	6.87	7.99	9.28	10.76	12.47	14.43	16.67	19.24	22.19	44.41	86.50	304.91
18	1.20	1.43	1.70	2.03	2.41	2.85	3.38	4.00	4.72	5.56	6.54	7.69	9.02	10.58	12.38	14.46	16.88	19.67	22.90	26.62	55.51	112.46	428.88
19	1.21	1.46	1.75	2.11	2.53	3.03	3.62	4.32	5.14	6.12	7.26	8.61	10.20	12.06	14.23	16.78	19.75	23.21	27.25	31.95	69.39	146.19	597.63
20	1.22	1.49	1.81	2.19	2.65	3.21	3.87	4.66	5.60	6.73	8.06	9.65	11.52	13.74	16.37	19.46	23.11	27.39	32.43	38.34	86.74	190.05	836.68
21	1.23	1.52	1.86	2.28	2.79	3.40	4.14	5.03	6.11	7.40	8.95	10.80	13.02	15.67	18.82	22.57	27.03	32.32	38.59	46.01	108.42	247.06	1,171.36
22	1.24	1.55	1.92	2.37	2.93	3.60	4.43	5.44	6.66	8.14	9.93	12.10	14.71	17.86	21.64	26.19	31.63	38.14	45.92	55.21	135.53	321.18	1,639.90
23	1.26	1.58	1.97	2.46	3.07	3.82	4.74	5.87	7.26	8.95	11.03	13.55	16.63	20.36	24.89	30.38	37.01	45.01	54.65	66.25	169.41	417.54	2,295.86
24	1.27	1.61	2.03	2.56	3.23	4.05	5.07	6.34	7.91	9.85	12.24	15.18	18.79	23.21	28.63	35.24	43.30	53.11	65.03	79.50	211.76	542.80	3,214.20
25	1.28	1.64	2.09	2.67	3.39	4.29	5.43	6.85	8.62	10.83	13.59	17.00	21.23	26.46	32.92	40.87	50.66	62.67	77.39	95.40	264.70	705.64	4,498.88
30	1.35	1.81	2.43	3.24	4.32	5.74	7.61	10.06	13.27	17.45	22.89	29.96	39.12	50.95	66.21	85.85	111.06	143.37	184.68	237.38	807.79	2,620.00	24,201.4
40	1.49	2.21	3.26	4.80	7.04	10.29	14.97	21.72	31.41	45.26	65.00	93.05	132.78	188.88	267.86	378.72	533.87	750.38	1,051.67	1,469.77	7,523.16	36,118.9	700,038
50	1.64	2.69	4.38	7.11	11.47	18.42	29.46	46.90	74.36	117.39	184.56	289.00	450.74	700.23	1,083.66	1,670.70	2,566.22	3,927.36	5,988.91	9,100.44	70,064.9	497,929	
60	1.82	3.28	5.89	10.52	18.68	32.99	57.95	101.26	176.03	304.48	524.06	897.60	1,530.05	2,595.92	4,384.00	7,370.20	12,335.4	20,555.1	34,105.0	56,347.5	662,530		

**Table 2: Present Value Interest Factor (PVIF) (\$1 at  $r^0$  for  $n$  periods)**  
 $PVIF = (1+r)^{-n}$  ;  $PV = FV (PVIF_{r,n})$

$n/r$	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%	13%	14%	15%	16%	17%	18%	19%	20%	25%	30%	40%	
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1	0.990	0.980	0.971	0.962	0.952	0.943	0.935	0.926	0.917	0.909	0.901	0.893	0.885	0.877	0.870	0.862	0.855	0.847	0.840	0.833	0.800	0.769	0.714	
2	0.980	0.961	0.943	0.925	0.907	0.890	0.873	0.857	0.842	0.826	0.812	0.797	0.783	0.769	0.756	0.743	0.731	0.718	0.706	0.694	0.640	0.592	0.510	
3	0.971	0.942	0.915	0.889	0.864	0.840	0.816	0.794	0.772	0.751	0.731	0.712	0.693	0.675	0.658	0.641	0.624	0.609	0.593	0.579	0.512	0.455	0.364	
4	0.961	0.924	0.888	0.855	0.823	0.792	0.763	0.735	0.708	0.683	0.659	0.636	0.613	0.592	0.572	0.552	0.534	0.516	0.499	0.482	0.410	0.350	0.260	
5	0.951	0.906	0.863	0.822	0.784	0.747	0.713	0.681	0.650	0.621	0.593	0.567	0.543	0.519	0.497	0.476	0.456	0.437	0.419	0.402	0.328	0.269	0.186	
6	0.942	0.888	0.837	0.790	0.746	0.705	0.666	0.630	0.596	0.564	0.535	0.507	0.480	0.456	0.432	0.410	0.390	0.370	0.352	0.335	0.262	0.207	0.133	
7	0.933	0.871	0.813	0.760	0.711	0.665	0.623	0.583	0.547	0.513	0.482	0.452	0.425	0.400	0.376	0.354	0.333	0.314	0.296	0.279	0.210	0.159	0.095	
8	0.923	0.853	0.789	0.731	0.677	0.627	0.582	0.540	0.502	0.467	0.434	0.404	0.376	0.351	0.327	0.305	0.285	0.266	0.249	0.233	0.168	0.123	0.068	
9	0.914	0.837	0.766	0.703	0.645	0.592	0.544	0.500	0.460	0.424	0.391	0.361	0.333	0.308	0.284	0.263	0.243	0.225	0.209	0.194	0.134	0.094	0.048	
10	0.905	0.820	0.744	0.676	0.614	0.558	0.508	0.463	0.422	0.386	0.352	0.322	0.295	0.270	0.247	0.227	0.208	0.191	0.176	0.162	0.107	0.073	0.035	
11	0.896	0.804	0.722	0.650	0.585	0.527	0.475	0.429	0.388	0.350	0.317	0.287	0.261	0.237	0.215	0.195	0.178	0.162	0.148	0.135	0.086	0.056	0.025	
12	0.887	0.788	0.701	0.625	0.557	0.497	0.444	0.397	0.356	0.319	0.286	0.257	0.231	0.208	0.187	0.168	0.152	0.137	0.124	0.112	0.069	0.043	0.018	
13	0.879	0.773	0.681	0.601	0.530	0.469	0.415	0.368	0.326	0.290	0.258	0.229	0.204	0.182	0.163	0.145	0.130	0.116	0.104	0.093	0.055	0.033	0.013	
14	0.870	0.758	0.661	0.577	0.505	0.442	0.388	0.340	0.299	0.263	0.232	0.205	0.181	0.160	0.141	0.125	0.111	0.099	0.088	0.078	0.044	0.025	0.009	
15	0.861	0.743	0.642	0.555	0.481	0.417	0.362	0.315	0.275	0.239	0.209	0.183	0.160	0.140	0.123	0.108	0.095	0.084	0.074	0.065	0.035	0.020	0.006	
16	0.853	0.728	0.623	0.534	0.458	0.394	0.339	0.292	0.252	0.218	0.188	0.163	0.141	0.123	0.107	0.093	0.081	0.071	0.062	0.054	0.028	0.015	0.005	
17	0.844	0.714	0.605	0.513	0.436	0.371	0.317	0.270	0.231	0.198	0.170	0.146	0.125	0.108	0.093	0.080	0.069	0.060	0.052	0.045	0.023	0.012	0.003	
18	0.836	0.700	0.587	0.494	0.416	0.350	0.296	0.250	0.212	0.180	0.153	0.130	0.111	0.095	0.081	0.069	0.059	0.051	0.044	0.038	0.018	0.009	0.002	
19	0.828	0.686	0.570	0.475	0.396	0.331	0.277	0.232	0.194	0.164	0.138	0.116	0.098	0.083	0.070	0.060	0.051	0.043	0.037	0.031	0.014	0.007	0.002	
20	0.820	0.673	0.554	0.456	0.377	0.312	0.258	0.215	0.178	0.149	0.124	0.104	0.087	0.073	0.061	0.051	0.043	0.037	0.031	0.026	0.012	0.005	0.001	
21	0.811	0.660	0.538	0.439	0.359	0.294	0.242	0.199	0.164	0.135	0.112	0.093	0.077	0.064	0.053	0.044	0.037	0.031	0.026	0.022	0.009	0.004	0.001	
22	0.803	0.647	0.522	0.422	0.342	0.278	0.226	0.184	0.150	0.123	0.101	0.083	0.068	0.056	0.046	0.038	0.032	0.026	0.022	0.018	0.007	0.003	0.001	
23	0.795	0.634	0.507	0.406	0.326	0.262	0.211	0.170	0.138	0.112	0.091	0.074	0.060	0.049	0.040	0.033	0.027	0.022	0.018	0.015	0.006	0.002	0.000	
24	0.788	0.622	0.492	0.390	0.310	0.247	0.197	0.158	0.126	0.102	0.082	0.066	0.053	0.043	0.035	0.028	0.023	0.019	0.015	0.013	0.005	0.002	0.000	
25	0.780	0.610	0.478	0.375	0.295	0.233	0.184	0.146	0.116	0.092	0.074	0.059	0.047	0.038	0.030	0.024	0.020	0.016	0.013	0.010	0.004	0.001	0.000	
30	0.742	0.552	0.412	0.308	0.231	0.174	0.131	0.099	0.075	0.057	0.044	0.033	0.026	0.020	0.015	0.012	0.009	0.007	0.005	0.004	0.001	0.000	0.000	
40	0.672	0.453	0.307	0.208	0.142	0.097	0.067	0.046	0.032	0.022	0.015	0.011	0.008	0.005	0.004	0.003	0.002	0.001	0.001	0.001	0.000	0.000	0.000	
50	0.608	0.372	0.228	0.141	0.087	0.054	0.034	0.021	0.013	0.009	0.005	0.003	0.002	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	
60	0.550	0.305	0.170	0.095	0.054	0.030	0.017	0.010	0.006	0.003	0.002	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	

**Table 3:** Future Value of an Annuity Interest Factor (FVIFA) (\$1 per period at r% for n periods)  
 $FVIFA = \frac{(1+r)^n - 1}{r}$  ;  $FVAN = PMT (FVIFA_{r,n})$

n/r	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	2.010	2.020	2.030	2.040	2.050	2.060	2.070	2.080	2.090	2.100	2.110	2.120
3	3.030	3.060	3.091	3.122	3.153	3.184	3.215	3.246	3.278	3.310	3.342	3.374
4	4.060	4.122	4.184	4.246	4.310	4.375	4.440	4.506	4.573	4.641	4.710	4.779
5	5.101	5.204	5.309	5.416	5.526	5.637	5.751	5.867	5.985	6.105	6.228	6.353
6	6.152	6.308	6.468	6.633	6.802	6.975	7.153	7.336	7.523	7.716	7.913	8.115
7	7.214	7.434	7.662	7.898	8.142	8.394	8.654	8.923	9.200	9.487	9.783	10.089
8	8.286	8.583	8.892	9.214	9.549	9.897	10.260	10.637	11.028	11.436	11.859	12.300
9	9.369	9.755	10.159	10.583	11.027	11.491	11.978	12.488	13.021	13.579	14.164	14.776
10	10.462	10.950	11.464	12.006	12.578	13.181	13.816	14.487	15.193	15.937	16.722	17.549
11	11.567	12.169	12.808	13.486	14.207	14.972	15.784	16.645	17.560	18.531	19.561	20.655
12	12.683	13.412	14.192	15.026	15.917	16.870	17.888	18.977	20.141	21.384	22.713	24.133
13	13.809	14.680	15.618	16.627	17.713	18.882	20.141	21.495	22.953	24.523	26.212	28.029
14	14.947	15.974	17.086	18.292	19.599	21.015	22.550	24.215	26.019	27.975	30.095	32.393
15	16.097	17.293	18.599	20.024	21.579	23.276	25.129	27.152	29.361	31.772	34.405	37.280
16	17.258	18.639	20.157	21.825	23.657	25.673	27.888	30.324	33.003	35.950	39.190	42.753
17	18.430	20.012	21.762	23.698	25.840	28.213	30.840	33.750	36.974	40.545	44.501	48.884
18	19.615	21.412	23.414	25.645	28.132	30.906	33.999	37.450	41.301	45.599	50.396	55.750
19	20.811	22.841	25.117	27.671	30.539	33.760	37.379	41.446	46.018	51.159	56.939	63.440
20	22.019	24.297	26.870	29.778	33.066	36.786	40.995	45.762	51.160	57.275	64.203	72.052
21	23.239	25.783	28.676	31.969	35.719	39.993	44.865	50.423	56.765	64.002	72.265	81.699
22	24.472	27.299	30.537	34.248	38.505	43.392	49.006	55.457	62.873	71.403	81.214	92.503
23	25.716	28.845	32.453	36.618	41.430	46.996	53.436	60.893	69.532	79.543	91.148	104.603
24	26.973	30.422	34.426	39.083	44.502	50.816	58.177	66.765	76.790	88.497	102.174	118.155
25	28.243	32.030	36.459	41.646	47.727	54.865	63.249	73.106	84.701	98.347	114.413	133.334
30	34.785	40.568	47.575	56.085	66.439	79.058	94.461	113.283	136.308	164.494	199.021	241.333
40	48.886	60.402	75.401	95.026	120.800	154.762	199.635	259.057	337.882	442.593	581.826	767.091
50	64.463	84.579	112.797	152.667	209.348	290.336	406.529	573.770	815.084	1,163.91	1,668.77	2,400.0
60	81.670	114.052	163.053	237.991	353.584	533.128	813.520	1,253.21	1,944.79	3,034.82	4,755.07	7,471.6

n/r	13%	14%	15%	16%	17%	18%	19%	20%	25%	30%	40%
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	2.130	2.140	2.150	2.160	2.170	2.180	2.190	2.200	2.250	2.300	2.400
3	3.407	3.440	3.473	3.506	3.539	3.572	3.606	3.640	3.813	3.990	4.360
4	4.850	4.921	4.993	5.066	5.141	5.215	5.291	5.368	5.766	6.187	7.104
5	6.480	6.610	6.742	6.877	7.014	7.154	7.297	7.442	8.207	9.043	10.946
6	8.323	8.536	8.754	8.977	9.207	9.442	9.683	9.930	11.259	12.756	16.324
7	10.405	10.730	11.067	11.414	11.772	12.142	12.523	12.916	15.073	17.583	23.853
8	12.757	13.233	13.727	14.240	14.773	15.327	15.902	16.499	19.842	23.858	34.395
9	15.416	16.085	16.786	17.519	18.285	19.086	19.923	20.799	25.802	32.015	49.153
10	18.420	19.337	20.304	21.321	22.393	23.521	24.709	25.959	33.253	42.619	69.814
11	21.814	23.045	24.349	25.733	27.200	28.755	30.404	32.150	42.566	56.405	98.739
12	25.650	27.271	29.002	30.850	32.824	34.931	37.180	39.581	54.208	74.327	139.235
13	29.985	32.089	34.352	36.786	39.404	42.219	45.244	48.497	68.760	97.625	195.929
14	34.883	37.581	40.505	43.672	47.103	50.818	54.841	59.196	86.949	127.913	275.300
15	40.417	43.842	47.580	51.660	56.110	60.965	66.261	72.035	109.687	167.286	386.420
16	46.672	50.980	55.717	60.925	66.649	72.939	79.850	87.442	138.109	218.472	541.988
17	53.739	59.118	65.075	71.673	78.979	87.068	96.022	105.931	173.636	285.014	759.784
18	61.725	68.394	75.836	84.141	93.406	103.740	115.266	128.117	218.045	371.518	1,064.7
19	70.749	78.969	88.212	98.603	110.285	123.414	138.166	154.740	273.556	483.973	1,491.6
20	80.947	91.025	102.444	115.380	130.033	146.628	165.418	186.688	342.945	630.165	2,089.2
21	92.470	104.768	118.810	134.841	153.139	174.021	197.847	225.026	429.681	820.215	2,925.9
22	105.491	120.436	137.632	157.415	180.172	206.345	236.438	271.031	538.101	1,067.3	4,097.2
23	120.205	138.297	159.276	183.601	211.801	244.487	282.362	326.237	673.626	1,388.5	5,737.1
24	136.831	158.659	184.168	213.978	248.808	289.494	337.010	392.484	843.033	1,806.0	8,033.0
25	155.620	181.871	212.793	249.214	292.105	342.603	402.042	471.981	1,054.8	2,348.8	11,247.2
30	293.199	356.787	434.745	530.312	647.439	790.948	966.712	1,181.9	3,227.2	8,730.0	60,501.1
40	1,013.7	1,342.0	1,779.1	2,360.8	3,134.5	4,163.2	5,529.8	7,343.9	30,088.7	120,393	
50	3,459.5	4,994.5	7,217.7	10,435.6	15,089.5	21,813.1	31,515.3	45,497.2	280,256		
60	11,761.9	18,535.1	29,220.0	46,057.5	72,555.0	114,190	179,495	281,733			

**Table 4:** Present Value of an Annuity Interest Factor (PVIFA) (\$1 per period at r% for n periods)  
 $PVIFA = \frac{1-(1+r)^{-n}}{r}$  ;  $PVAN = PMT (PVIFA_{r,n})$

n/r	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%
1	0.990	0.980	0.971	0.962	0.952	0.943	0.935	0.926	0.917	0.909	0.901	0.893
2	1.970	1.942	1.913	1.886	1.859	1.833	1.808	1.783	1.759	1.736	1.713	1.690
3	2.941	2.884	2.829	2.775	2.723	2.673	2.624	2.577	2.531	2.487	2.444	2.402
4	3.902	3.808	3.717	3.630	3.546	3.465	3.387	3.312	3.240	3.170	3.102	3.037
5	4.853	4.713	4.580	4.452	4.329	4.212	4.100	3.993	3.890	3.791	3.696	3.605
6	5.795	5.601	5.417	5.242	5.076	4.917	4.767	4.623	4.486	4.355	4.231	4.111
7	6.728	6.472	6.230	6.002	5.786	5.582	5.389	5.206	5.033	4.868	4.712	4.564
8	7.652	7.325	7.020	6.733	6.463	6.210	5.971	5.747	5.535	5.335	5.146	4.968
9	8.566	8.162	7.786	7.435	7.108	6.802	6.515	6.247	5.995	5.759	5.537	5.328
10	9.471	8.983	8.530	8.111	7.722	7.360	7.024	6.710	6.418	6.145	5.889	5.650
11	10.368	9.787	9.253	8.760	8.306	7.887	7.499	7.139	6.805	6.495	6.207	5.938
12	11.255	10.575	9.954	9.385	8.863	8.384	7.943	7.536	7.161	6.814	6.492	6.194
13	12.134	11.348	10.635	9.986	9.394	8.853	8.358	7.904	7.487	7.103	6.750	6.424
14	13.004	12.106	11.296	10.563	9.899	9.295	8.745	8.244	7.786	7.367	6.982	6.628
15	13.865	12.849	11.938	11.118	10.380	9.712	9.108	8.559	8.061	7.606	7.191	6.811
16	14.718	13.578	12.561	11.652	10.838	10.106	9.447	8.851	8.313	7.824	7.379	6.974
17	15.562	14.292	13.166	12.166	11.274	10.477	9.763	9.122	8.544	8.022	7.549	7.120
18	16.398	14.992	13.754	12.659	11.690	10.828	10.059	9.372	8.756	8.201	7.702	7.250
19	17.226	15.678	14.324	13.134	12.085	11.158	10.336	9.604	8.950	8.365	7.839	7.366
20	18.046	16.351	14.877	13.590	12.462	11.470	10.594	9.818	9.129	8.514	7.963	7.469
21	18.857	17.011	15.415	14.029	12.821	11.764	10.836	10.017	9.292	8.649	8.075	7.562
22	19.660	17.658	15.937	14.451	13.163	12.042	11.061	10.201	9.442	8.772	8.176	7.645
23	20.456	18.292	16.444	14.857	13.489	12.303	11.272	10.371	9.580	8.883	8.266	7.718
24	21.243	18.914	16.936	15.247	13.799	12.550	11.469	10.529	9.707	8.985	8.348	7.784
25	22.023	19.523	17.413	15.622	14.094	12.783	11.654	10.675	9.823	9.077	8.422	7.843
30	25.808	22.396	19.600	17.292	15.372	13.765	12.409	11.258	10.274	9.427	8.694	8.055
40	32.835	27.355	23.115	19.793	17.159	15.046	13.332	11.925	10.757	9.779	8.951	8.244
50	39.196	31.424	25.730	21.482	18.256	15.762	13.801	12.233	10.962	9.915	9.042	8.304
60	44.955	34.761	27.676	22.623	18.929	16.161	14.039	12.377	11.048	9.967	9.074	8.324

n/r	13%	14%	15%	16%	17%	18%	19%	20%	25%	30%	40%
1	0.885	0.877	0.870	0.862	0.855	0.847	0.840	0.833	0.800	0.769	0.714
2	1.668	1.647	1.626	1.605	1.585	1.566	1.547	1.528	1.440	1.361	1.224
3	2.361	2.322	2.283	2.246	2.210	2.174	2.140	2.106	1.952	1.816	1.589
4	2.974	2.914	2.855	2.798	2.743	2.690	2.639	2.589	2.362	2.166	1.849
5	3.517	3.433	3.352	3.274	3.199	3.127	3.058	2.991	2.689	2.436	2.035
6	3.998	3.889	3.784	3.685	3.589	3.498	3.410	3.326	2.951	2.643	2.168
7	4.423	4.288	4.160	4.039	3.922	3.812	3.706	3.605	3.161	2.802	2.263
8	4.799	4.639	4.487	4.344	4.207	4.078	3.954	3.837	3.329	2.925	2.331
9	5.132	4.946	4.772	4.607	4.451	4.303	4.163	4.031	3.463	3.019	2.379
10	5.426	5.216	5.019	4.833	4.659	4.494	4.339	4.192	3.571	3.092	2.414
11	5.687	5.453	5.234	5.029	4.836	4.656	4.486	4.327	3.656	3.147	2.438
12	5.918	5.660	5.421	5.197	4.988	4.793	4.611	4.439	3.725	3.190	2.456
13	6.122	5.842	5.583	5.342	5.118	4.910	4.715	4.533	3.780	3.223	2.469
14	6.302	6.002	5.724	5.468	5.229	5.008	4.802	4.611	3.824	3.249	2.478
15	6.462	6.142	5.847	5.575	5.324	5.092	4.876	4.675	3.859	3.268	2.484
16	6.604	6.265	5.954	5.668	5.405	5.162	4.938	4.730	3.887	3.283	2.489
17	6.729	6.373	6.047	5.749	5.475	5.222	4.990	4.775	3.910	3.295	2.492
18	6.840	6.467	6.128	5.818	5.534	5.273	5.033	4.812	3.928	3.304	2.494
19	6.938	6.550	6.198	5.877	5.584	5.316	5.070	4.843	3.942	3.311	2.496
20	7.025	6.623	6.259	5.929	5.628	5.353	5.101	4.870	3.954	3.316	2.497
21	7.102	6.687	6.312	5.973	5.665	5.384	5.127	4.891	3.963	3.320	2.498
22	7.170	6.743	6.359	6.011	5.696	5.410	5.149	4.909	3.970	3.323	2.498
23	7.230	6.792	6.399	6.044	5.723	5.432	5.167	4.925	3.976	3.325	2.499
24	7.283	6.835	6.434	6.073	5.746	5.451	5.182	4.937	3.981	3.327	2.499
25	7.330	6.873	6.464	6.097	5.766	5.467	5.195	4.948	3.985	3.329	2.499
30	7.496	7.003	6.566	6.177	5.829	5.517	5.235	4.979	3.995	3.332	2.500
40	7.634	7.105	6.642	6.233	5.871	5.548	5.258	4.997	3.999	3.333	2.500
50	7.675	7.133	6.661	6.246	5.880	5.554	5.262	4.999	4.000	3.333	2.500
60	7.687	7.140	6.665	6.249	5.882	5.555	5.263	5.000	4.000	3.333	2.500

# Arithmetic Progression and Geometric Progression

## 1.3

### Arithmetic Progression:

For any series with ' $n$ ' terms, if the difference between two consecutive terms is constant throughout the series, the series is termed as Arithmetic Progression (A. P.). The Difference between two consecutive terms can be either positive or negative.

Difference between two consecutive terms is represented by " $d$ " and known as Common Difference.

### Formula for calculating the value Common Difference (D) of an A.P

We have " **$d$** " = 2nd Term – 1st Term or (Any term – its preceding term)

The value of " $d$ " is constant throughout the Series. In other words, " $d$ " calculated from any two consecutive terms will be equal throughout

For example,

$$d_1 = 5^{\text{th}} \text{ term} - 4^{\text{th}} \text{ term}$$

$$d_2 = 8^{\text{th}} \text{ term} - 7^{\text{th}} \text{ term}$$

Both,  $d_1$  and  $d_2$  will be equal for any given A. P.

If the common difference is Positive ( $d > 0$ ), the terms of the series will be in ascending order. In other words, the 1<sup>st</sup> term will be smaller than the 2<sup>nd</sup> term and so on.

For example,

Series: 1, 3, 5, 7, 9 has " $d$ " = 2 { $5-3 = 2$ ;  $3-1 = 2$ }

### Illustration 57.

A given series is: 26, 29, 32, 35, 38, 41 Identify the type of Series. Also calculate the values of " $d$ " and ' $a$ '.

### Solution:

From the given series it can be inferred that any term of the series is obtained by adding a constant to the preceding term. So the type of Series is A. P.

Values of " $d$ " and ' $a$ ':

$$'a' - \text{First Term of the Series} = 26.$$

" $d$ " – Common Difference between any two consecutive terms of the series

$$"d" = 2^{\text{nd}} \text{ term} - 1^{\text{st}} \text{ term} = 29 - 26 = 3$$

So Type of Series- A. P. with 1<sup>st</sup> Term ( $a$ ) = 26 and Common Difference ( $d$ ) = 3.

**Illustration 58.**

Form an A.P. with the value of “ $d$ ” = 6, ‘ $a$ ’ = – 13. Determine first 5 terms.

**Solution:**

The following values of “ $d$ ” and ‘ $a$ ’ are given,

‘ $a$ ’ – First Term of the Series = – 13.

‘ $d$ ’ – Common Difference between any two consecutive terms of the series = 6

The first 5 terms of the Series would be determined as follows:

– 13, (– 13+6), (– 13+6+6), (– 13+6+6+6), (– 13+6+6+6+6)

A. P. Series with first 5 terms: – 13, – 7, – 1, 5, 11

**Formula for determining the value of  $n^{\text{th}}$  term ( $t_n$ ) of an A.P.**

$$t_n = a + (n - 1) \times d$$

Where,

$t_n$  = Value of the  $n^{\text{th}}$  term

$a$  = First Term of the Series

$n$  = Number of terms in the Series

$d$  = Common Difference between any two consecutive terms of the Series

The above mentioned formula can be taken into consideration for the following:

- (i) To find the number of terms of the Series.
- (ii) To Find the Common Difference between two consecutive terms of the Series.
- (iii) To Find the First Term of the Series.
- (iv) To Find the Value of the  $n^{\text{th}}$  term of the Series.
- (v) To Find the Value of the Last Term of the Series, if value of ‘ $n$ ’ is given.

For the above illustration  $a = -13$  and  $d = 6$

1<sup>st</sup> term =  $a = -13$  (given)

Using formula  $t_n = a + (n - 1)d$

2<sup>nd</sup> term =  $t_2 = -13 + (2 - 1) \times 6 = -13 + 1 \times 6 = -13 + 6 = -7$

3<sup>rd</sup> term =  $t_3 = -13 + (3 - 1) \times 6 = -13 + 2 \times 6 = -13 + 12 = -1$

4<sup>th</sup> term =  $t_4 = -13 + (4 - 1) \times 6 = -13 + 3 \times 6 = -13 + 18 = 5$

5<sup>th</sup> term =  $t_5 = -13 + (5 - 1) \times 6 = -13 + 4 \times 6 = -13 + 24 = 11$

A. P. Series with first 5 terms: – 13, – 7, – 1, 5, 11.

If the Common Difference is Negative ( $d < 0$ ), the terms of the series will be in descending order. In other words,

the 1<sup>st</sup> term will be larger than the 2<sup>nd</sup> next term and so on.

For example,

Series: 9, 7, 5, 3, 1 has  $d = -2$  { $5 - 7 = -2$ ;  $7 - 9 = -2$ }

### Illustration 59.

A given series is:  $a + 14, a + 12, a + 10, a + 8, a + 6$  Identify the type of Series. Also calculate the values of 'a' and  $d$ .

#### Solution:

Since from the Series it is observed that Subtraction of an constant being made to obtain the consecutive terms, the type of Series is A. P.

Values of  $a$  &  $d$ :

' $a$ ' – First Term of the Series =  $a + 14$ .

' $d$ ' – Common Difference between two consecutive terms of the series

' $d$ ' = 2<sup>nd</sup> term – 1<sup>st</sup> term =  $(a + 12) - (a + 14) = a + 12 - a - 14 = 12 - 14 = -2$

So Type of Series- A. P. with 1<sup>st</sup> Term ( $a$ ) =  $a + 14$  and Common Difference ( $d$ ) =  $-2$ .

### Formula for Calculating the Sum of the terms ( $S_n$ ) for any A.P.

$$S_n = \frac{n}{2} \times \{2a + (n-1) \times d\}$$

Where

$S_n$  = Sum upto  $n$  terms of the series

Number of terms in the series =  $n$

1<sup>st</sup> Term of the series =  $a$

Common Difference between any two consecutive terms of the series =  $d$

### Illustration 60.

Find the sum of the first 10 terms for the series: 12, 15, 18, 21, 24.

#### Solution:

For calculating the sum of first 10 terms of the given series: 12, 15, 18, 21, 24, the following formula is used.

$$S_n = \frac{n}{2} \times \{2a + (n-1) \times d\}$$

$n$  = Number of terms in the series = 10

$a$  = 1<sup>st</sup> Term of the series = 12

$d$  = Common Difference between any two consecutive terms of the series = 2<sup>nd</sup> term – 1<sup>st</sup> term =  $15 - 12 = 3$

$$S_{10} = (10/2) \times \{2 \times 12 + (10-1) \times 3\}$$

$$\text{Or, } S_{10} = 5 \times \{24 + 9 \times 3\}$$

$$\text{Or, } S_{10} = 5 \times \{24 + 27\} = 5 \times 51 = 255.$$

### Geometric Progression

For any series with ' $n$ ' terms, if the ratio between two consecutive terms is constant throughout the series, the series is termed as Geometric Progression (G. P.). The Common Ratio between two consecutive terms can be either positive or negative.

Common Ratio between two consecutive terms is represented by ' $r$ '.

### Formula for calculating the value of Common Ratio ( $r$ ) of a G.P.

$$\text{We have } r = \frac{2^{\text{nd}} \text{ Term}}{1^{\text{st}} \text{ Term}} \text{ Or } \frac{\text{Any Term}}{\text{Previous Term}}$$

The value of  $r$  is constant throughout the Series. In other words, ' $r$ ' calculated from any two consecutive terms will be throughout.

For example,

$$r_1 = 6^{\text{th}} \text{ term} / 5^{\text{th}} \text{ term}$$

$$r_2 = 9^{\text{th}} \text{ term} / 8^{\text{th}} \text{ term}$$

Both,  $r_1$  and  $r_2$  will be equal for any given G. P.

If the Common Ratio is greater than 1 the terms of the series will be in ascending order.

For example,

$$\text{Series: } 1, 3, 9, 27, 81 \text{ has } 'r' = 3 \{9/3 = 3; 3/1 = 3\}$$

### Illustration 61.

A series is given as: 7, 14, 28, 56, 112 Identify the type of Series. Also calculate the values of ' $a$ ' and ' $r$ '.

#### Solution:

Since from the Series it is observed that there exist a Common Ratio between the consecutive terms, the type of Series is G. P.

Values of ' $a$ ' and ' $r$ ':

$$'a' - \text{First Term of the Series} = 7.$$

' $r$ ' – Common Ratio between any two consecutive terms of the series

$$'r' = 2^{\text{nd}} \text{ term} / 1^{\text{st}} \text{ term} = 14 / 7 = 2$$

So the type of Series is G. P. with 1<sup>st</sup> Term ( $a$ ) = 7 and Common Ratio ( $r$ ) = 2.

### Illustration 62.

For the given series: 1875, –375, 75, –15, 3, find out the values of ' $a$ ' and ' $r$ '. Also identify the type of Series.

**Solution:**

Since the Series has a Common Ratio between the consecutive terms, the type of Series is G. P.

Values of 'a' and 'r':

'a' – First Term of the Series = 1875.

'r' – Common Ratio between any two consecutive terms of the series

'r' = 2<sup>nd</sup> term / 1<sup>st</sup> term =  $-375 / 1875 = -1/5$

Type of Series- G. P. with 1<sup>st</sup> Term (a) = 1875 and Common Ratio (r) =  $-1/5$ .

**Formula for determining the value of n<sup>th</sup> term ( $t_n$ ) of a G.P**

$$t_n = ar^{n-1}$$

Where,

$t_n$  = Value of the n<sup>th</sup> term

a = First Term of the Series

n = Number of terms in the Series

r = Common Ratio between two consecutive terms of the Series

The above mentioned formula can be taken into consideration for the following:

- (i) To find the number of terms of the Series.
- (ii) To Find the Common Ratio between two consecutive terms of the Series.
- (iii) To Find the First Term of the Series.
- (iv) To Find the Value of the n<sup>th</sup> term of the Series.
- (v) To Find the Value of the Last Term of the Series, if value of 'n' is given.

**Illustration 63.**

Determine 9<sup>th</sup> term for the series: 11, 33, 99, 297, 891.

**Solution:**

We know that  $t_n = ar^{n-1}$

a = First Term of the Series = 11

n = Number of terms in the Series = 9

r = Common Ratio between two consecutive terms of the Series

r = 2<sup>nd</sup> term / 1<sup>st</sup> term =  $33/11 = 3$

$$t_n = a \times r^{(n-1)}$$

$$\text{Or, } t_9 = 11 \times 3^{(9-1)}$$

$$\text{Or, } t_9 = 11 \times 3^{(8)} = 72171$$

**Formula for Calculating the Sum of the terms ( $S_n$ ) for any G. P.**

$$S_n = a \frac{r^n - 1}{r - 1}, \text{ When } r > 1$$

$$S_n = a \frac{1 - r^n}{1 - r}, \text{ When } r < 1$$

Where

$S_n$  = Sum upto  $n$  terms of the series

$n$  = number of terms in the series

$a$  = 1st Term of the series

$r$  = Common Ratio between any two consecutive terms of the series

**For a G.P with infinite terms, the formula for Summation is  $S = \frac{a}{1-r}$  when  $r < 1$**

**Illustration 64.**

Find the sum of the first 9 terms for the series: 5, 10, 20, 40, 80.

**Solution:**

$n$  = number of terms in the series = 9

$a$  = 1<sup>st</sup> Term of the series = 5

$r$  = Common Ratio between any two consecutive terms of the series

$r = 2^{\text{nd}} \text{ term} / 1^{\text{st}} \text{ term} = 10/5 = 2 > 1$

We know that  $S_n = a \frac{r^n - 1}{r - 1}$  Since  $r > 1$

$$S_9 = 5 \times \{(2^9 - 1) / (2 - 1)\}$$

$$= 5 \times \{(2^9 - 1) / 1\}$$

$$= 5 \times (512 - 1)$$

$$= 5 \times 511 = 2555$$

**Illustration 65.**

Find the sum of the first 10 terms for the series: 48, 24, 12, 6, 3.

**Solution:**

$n$  = number of terms in the series = 10

$a$  = 1<sup>st</sup> Term of the series = 48

$r$  = Common Ratio between any two consecutive terms of the series

$r = 2^{\text{nd}} \text{ term} / 1^{\text{st}} \text{ term} = 24/48 = 1/2 < 1$

We know that  $S_n = a \frac{1-r^n}{1-r}$  Since  $r < 1$

$$\begin{aligned} S_{10} &= 48 \times \{1 - (1/2)^{10}\} / (1 - 1/2) \\ &= 48 \times \{1 - (1/2)^{10}\} / (1/2) \\ &= 48 \times \{1 - (1/2)^{10}\} \times 2 \\ &= 96 \times \{1 - (1/2)^{10}\} \\ &= 96 \times (1 - 0.00098) \\ &= 96 \times 0.99902 = 95.9059 \end{aligned}$$

**Illustration 66.**

Find nth term of the sequence -1, 2, -4, 8,.....

**Solution:**

$$t_n = (-1)^n 2^{n-1}.$$

**Illustration 67.**

In A.P if  $S_n = n^2 + 2n$ , find  $t_n$ .

**Solution:**

$$\begin{aligned} t_n &= S_n - S_{n-1} \\ &= (n^2 + 2n) - [(n-1)^2 + 2(n-1)] = 2n + 1. \end{aligned}$$

**Illustration 68.**

If  $x = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots \infty$

$Y = 1 + \frac{1}{4} + \frac{1}{4^2} + \dots \infty$ , find  $xy$ .

**Solution:**

$$x = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}, \quad y = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

$$xy = \frac{3}{2} \times \frac{4}{3} = 2.$$

**Illustration 69.**

Find sum of series  $1 - 1 + 1 - 1 \dots \infty$

**Solution:**

$$\begin{aligned} S &= 1 - 1 + 1 - 1 \dots \infty \\ &= \frac{1}{1 - (-1)} = \frac{1}{2}. \quad (a = 1, r = -1) \end{aligned}$$

**Illustration 70.**

8th term of an A.P is 15, then find sum of 15 term.

**Solution:**

$$\begin{aligned}t_8 &= a + 7d = 15 \\S_{15} &= \frac{15}{2} [2a + (15-1)d] \\&= \frac{15}{2} \times 2(a+7d) \\&= 15 \times 15 = 225\end{aligned}$$

**Illustration 71.**

In an A.P, if  $S_n = 3n^2 - n$  & the common difference in 6, then find 1st term.

**Solution:**

$$\begin{aligned}S^n &= 3n^2 - n \\S_1 &= 3 \times 1 - 1 = 2 = \text{1st term.}\end{aligned}$$

**Illustration 72.**

Find 1st term of a G.P, where 2nd term is 2 and sum of infinite term is 8.

**Solution:**

$$\begin{aligned}t_2 = ar = 2 \quad \text{or} \quad r &= \frac{2}{a} \\S_n = \frac{a}{1-r} = \frac{a}{1-\frac{2}{a}} &= 8 \\ \text{Or,} \quad a = 8\left(1 - \frac{2}{a}\right) \quad \text{Or,} \quad a &= 8 \frac{(a-2)}{a} \\ \text{Or,} \quad a^2 = 8(a-2) \\ \text{Or,} \quad a^2 = 8a-16 \\ \text{Or,} \quad a^2-8a+16 &= 0 \\ \text{Or,} \quad (a-4)^2 = 0 \quad \text{or} \quad a-4 &= 0 \quad \text{or} \quad a = 4.\end{aligned}$$

**Illustration 73.**

There are 3 nos a, b, c are in A.P. find a - b + c.

**Solution:**

$$\begin{aligned}a = 1, b = 2, c = 3 \\a-b+c = 1-2+3 &= 2.\end{aligned}$$

**Illustration 74.**

Sum of 3rd and 9th term of an AP is 8. Find the sum of 1st 11 terms.

**Solution:**

$$t_3 + t_9 = 8$$

$$a + 2d + a + 8d = 8$$

or,  $2a + 10d = 8$

$$S_{11} = \frac{11}{2}(2a + 10d) = \frac{11}{2} \times 8 = 44$$

**Illustration 75.**

Find sum of all natural nos. between 100 and 1000, which are multiple of 5.

**Solution:**

$$S = 105 + \dots + 995.$$

$$\text{No. of terms} = n = \frac{995 - 105}{5} + 1 = 179$$

$$S_n = \frac{n}{2}(a + l) = \frac{179}{2}(105 + 995) = 98,450.$$

**EXERCISE WITH Hints:**

1. If  $m^{\text{th}}$  term of an AP is  $\frac{1}{n}$  and  $n^{\text{th}}$  term is  $\frac{1}{m}$ , and  $m \neq n$  then show that the first term is equal to the common difference

**Hints:**

Let  $a$  denote the first term and  $d$  the common difference. So by question we can write

$$t_m = \frac{1}{n} = a + (m - 1)d \text{ \& } t_n = \frac{1}{m} = a + (n - 1)d \text{ [} a = \text{1st term and } d = \text{Common difference]}$$

$$\text{Now } \frac{1}{n} - \frac{1}{m} = (m - 1)d - (n - 1)d$$

$$\text{Or, } \frac{m - n}{mn} = md - d - nd + d$$

$$\text{Or, } \frac{m - n}{mn} = (m - n)d$$

$$\text{Or, } d = \frac{1}{mn}$$

$$\text{Now from the expression for } t_m \text{ we can write } a = \frac{1}{n} - (m - 1)d = \frac{1}{n} - (m - 1)\frac{1}{mn} = \frac{1}{n} - \frac{m}{mn} + \frac{1}{mn} = \frac{1}{mn} = d \text{ (Proved)}$$

2. If  $a' = 1 + c^a + c^{2a} + c^{3a} + \dots \infty$  ( $0 < c < 1$ )

$$\text{And } b' = 1 + c^b + c^{2b} + c^{3b} + \dots \infty \text{ (} 0 < c < 1 \text{)}$$

$$\text{Then prove that } c = \frac{(a' - 1)^{\frac{1}{a}}}{a'} = \frac{(b' - 1)^{\frac{1}{b}}}{b'}$$

**Hints:**

$$\text{By question we can write } a' = 1 + c^a + c^{2a} + c^{3a} + \dots \infty = \frac{1}{1 - c^a} \text{ [from sum of infinite GP series]}$$

Similarly  $b' = 1 + c^b + c^{2b} + c^{3b} + \dots \dots \dots \infty = \frac{1}{1 - c^b}$

Now  $\frac{(a' - 1)^{\frac{1}{a}}}{a'} = \frac{\left(\frac{1}{1 - c^a} - 1\right)^{\frac{1}{a}}}{\frac{1}{1 - c^a}} = \frac{\left(\frac{1 - 1 + c^a}{1 - c^a}\right)^{\frac{1}{a}}}{\frac{1}{1 - c^a}} = \frac{(c^a)^{\frac{1}{a}}}{1} = c$

Similarly  $\frac{(b' - 1)^{\frac{1}{b}}}{b'} = c$

3. If *p*th, *q*th, *r*th terms of an AP are *a*, *b*, *c* respectively. Show that  $(q - r)a + (r - p)b + (p - q)c = 0$

**Hints:**

Let 1st term = *A* and Common difference = *d*

By question we can write

$a = A + (p - 1)d \dots \dots \dots (i)$

$b = A + (q - 1)d \dots \dots \dots (ii)$

$c = A + (r - 1)d \dots \dots \dots (iii)$

By (i) × (q - r), (ii) × (r - p) & (iii) × (p - q) and subsequently adding we get

$$\begin{aligned} & a(q - r) + b(r - p) + c(p - q) \\ &= A(q - r + r - p + p - q) \\ & \quad + d[(p - 1)(q - r) + (q - 1)(r - p) + (r - 1)(p - q)] \\ &= 0 + d[pq - q - pr + r + qr - r - pq + p + pr - p - qr + q] \\ &= 0 + 0 = 0 \end{aligned}$$

4. If  $x = a + \frac{a}{r} + \frac{a}{r^2} + \dots \dots \dots \infty$ ,  $y = b - \frac{b}{r} + \frac{b}{r^2} - \dots \dots \dots \infty$ , &  $z = c + \frac{c}{r^2} + \frac{c}{r^4} - \dots \dots \dots \infty$

Show that  $\frac{xy}{z} = \frac{ab}{c}$

**Hints:**

By applying sum of infinite G.P series formula

By question  $x = \frac{a}{1 - \frac{1}{r}} = \frac{ar}{r - 1}$

Similarly  $y = \frac{br}{r + 1}$  &  $z = \frac{cr^2}{r^2 - 1}$

Now  $\frac{xy}{z} = \frac{\frac{abr^2}{r^2 - 1}}{\frac{cr^2}{r^2 - 1}} = \frac{ab}{c}$

5. The rate of monthly salary for a person is increased annually in AP. It is known that he was drawing ₹400 a month during the 11<sup>th</sup> year of his service and ₹760 during the 29<sup>th</sup> year. Find his starting salary and the rate of annual increment. What should be his salary at the time of retirement just on the completion of 36<sup>th</sup> years of service?

**Hints:**

Let the salary per month in the 1<sup>st</sup> year is  $x$  and in every year it increased by an amount  $d$ .

So by question we can write:

$$t_{11} = x + (11 - 1)d = 400 \quad \dots \dots \dots (1)$$

$$t_{29} = x + (29 - 1)d = 760 \quad \dots \dots \dots (2)$$

So from (1) & (2) above we can write

$$x + 10d = 400 \quad \dots \dots \dots (3)$$

and

$$x + 28d = 760 \quad \dots \dots \dots (4)$$

By (4) – (3) we get  $18d = 360$ . So  $d = ₹20$ .

Putting this value of  $d$  in equation (3) we get  $x = 400 - 10 \times 20 = ₹200$

We have to find out  $t_{36} = x + (36 - 1)d = 200 + 35 \times 20 = ₹900$

6. A man secures an interest free loan of ₹14,500 from a friend and agrees to repay it in 10 instalments. He pays ₹1,000 as first instalment and then increases each instalment by equal amount over the preceding instalment. What will be his last instalment?

**Hints:**

Let the first instalment is  $A$  and equal increased amount is  $d$

So payment of instalments will form a series like

$$A + (A + d) + (A + 2d) + \dots \dots \dots 10 \text{ terms} = 14500$$

Now by question  $A = 1000$

$$\text{We can write } S_{10} = 14500 = \frac{10}{2} [2 \times 1000 + (10 - 1)d]$$

$$\text{Or, } 2900 = [2000 + 9d]$$

$$\text{Or, } 2900 - 2000 = 9d$$

$$\text{So, } d = 100$$

We have to find out the last instalment. That is we have to find the last term of the series.

$$\text{So we can write } t_{10} = A + 9d = 1000 + 9 \times 100 = ₹1900$$

7. Find the sum of the first 15 terms of a GP given by 1,  $-1/2$ ,  $1/4$ ,  $-1/8$ .....

**Hints:**

Here  $a = 1$ ,  $r = \frac{-1/2}{1} = \frac{1/4}{-1/2} = \frac{-1/8}{1/4} = -\frac{1}{2}$ ,  $n = 15$

So  $S_n = \frac{a(1-r^n)}{1-r} = \frac{1[1-(-1/2)^{15}]}{1-(-1/2)} = \frac{(2^{15}+1)^{15}}{1+1/2} = \frac{2^{15}+1}{3 \times 2^{14}} = \frac{10923}{16384}$

8. Evaluate the recurring decimal 0.19

**Hints:**

Now  $0.19 = 0.1 + 0.09 + 0.009 + 0.0009 + \dots$

$= \frac{1}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \dots$

$= \frac{1}{10} + \frac{9}{10^2} \times \left[ 1 + \frac{1}{10} + \frac{1}{10^2} \dots \dots \right]$

$= \frac{1}{10} + \frac{9}{10^2} \left[ \frac{1}{1 - \frac{1}{10}} \right]$  (Since the sum of the terms of an infinite G.P.  $= \frac{a}{1-r}$ , when  $r < 1$ )

$= \frac{1}{10} + \frac{9}{10^2} \times \frac{10}{9} = \frac{2}{10} = \frac{1}{5}$

9. If  $a, b, c, d$  are in GP then

- (a)  $a^2 + b^2, b^2 + c^2, c^2 + d^2$  are in GP
- (b)  $a^2 - b^2, b^2 - c^2, c^2 - d^2$  are in GP
- (c)  $a^2b^2, b^2c^2, c^2d^2$  are in GP
- (d)  $\frac{a^2}{b^2}, \frac{b^2}{c^2}, \frac{c^2}{d^2}$  are in GP

**Hints:**

We have  $b = ak, c = bk, d = ck$ . i.e.  $b = ak, c = ak^2$  &  $d = ak^3$ . Hence the answer is (b)

10. MR X arranges to pay off a debt of ₹9600 in 48 annual instalments which form an AP. When 40 of these instalments are paid, MR X becomes insolvent with Rs.2400 still remains unpaid. Then value of second instalment is

- (a) ₹97.50
- (b) ₹92.50
- (c) ₹82.50
- (d) ₹87.50

**Hints:**

We can write  $9600 = \frac{48}{2}[2a + (48-1)d]$  &  $(9600 - 2400) = \frac{40}{2}[2a + (40-1)d]$  & hence the answer is (d)

11. An enterprise produced 600 units in the 3<sup>rd</sup> year of its existence and 700 units in its 7<sup>th</sup> year. If production is maintained as an AP then production in the first year is

- (a) 580
- (b) 575
- (c) 550
- (d) 590

**Hints:**

By question  $600 = a + 2d$  &  $700 = a + 6d$  and hence the answer is (c)

12. The sum of first 6 terms of a GP is 9 times the sum of first three terms. The common ratio is

- (a) 3
- (b)  $\frac{1}{3}$
- (c) 1
- (d) 2

**Hints:**

$\frac{a(r^6 - 1)}{r - 1} = 9 \frac{a(r^3 - 1)}{r - 1}$ . Hence the answer is (d)

13. Find sum of all natural numbers from 100 to 300 which are divisible by 4

- (a) 10900
- (b) 10200
- (c) 10000
- (d) 11000

**Hints:**

Series is  $100 + 104 + 108 + \dots + 300$ . So  $300 = 100 + (n-1)4$ , Hence the answer is (b)

14. If you insert 9 arithmetic means between  $\frac{1}{4}$  and  $(-\frac{39}{4})$ , then 1<sup>st</sup> arithmetic mean is

- (a)  $-\frac{1}{5}$
- (b)  $-\frac{1}{8}$

(c) 0

(d)  $-\frac{3}{4}$ **Hints:**

9 arithmetic means are inserted between  $\frac{1}{4}$  &  $\frac{-39}{4}$  i.e. we have 11 terms on the whole. So  $-\frac{39}{4} = \frac{1}{4} + (11-1)d$ . Hence the answer is (d)

15. A man saved ₹16500 in 10 years. In each year after the first he saved ₹100 more than he did in the preceding year. In the first year his savings was

(a) ₹1000

(b) ₹1000

(c) ₹1200

(d) ₹800

**Hints:**

$16500 = \frac{10}{2} [2 \times a + 9 \times 100]$ , Hence the answer is (c)

16. A moneylender lends ₹1000 and charges an overall interest of ₹140. He recovers the loan and interest by 12 monthly instalments each less by ₹10 than the preceding one. The amount of the first instalment is

(a) ₹175

(b) ₹300

(c) ₹150

(d) ₹400

**Hints:**

$1000 + 140 = \frac{12}{2} [2 \times a + 11 \times (-)10]$ , Find 'a' to get the answer (c)

17. Sum of the series  $r^2 - 2r + 2$ ,  $r^2 - 2r + 3$ ,  $r^2 - 2r + 4$  up to  $(2r - 1)$  terms is

(a)  $4r^3 + 2r^2 + r + 3$ (b)  $r^3 + 5r^2 - 7r - 1$ 

(c) 1

(d)  $2r^3 - 3r^2 + 3r - 1$ **Hints:**

The given series is AP with c.d. = 1 and hence the answer is (d)

18. If  $(1+1+3+5+\text{up to } k \text{ terms}) - t_k = 0$ , then  $t_k$  is equal to

- (a)  $k^2 - 2k + 2$
- (b)  $k^2$
- (c)  $2k^2 - 5k + 3$
- (d)  $k$

**Hints:**

$(1 + 1 + 3 + 5 + \text{upto } k \text{ terms}) - t_k = 0$ . So  $t_k = (1 + 1 + 3 + 5 + \text{upto } k \text{ terms})$  Or,  $t_k = 1 + [1 + 3 + 5 + \text{upto } (k - 1) \text{ terms}]$  and the bracketed portion is an A.P. Hence the answer is (a)

19. The sum of  $\frac{3}{\sqrt{5}}, \frac{4}{\sqrt{5}}, \sqrt{5} \dots \dots \dots$  to 25 terms is

- (a)  $\frac{1}{\sqrt{5}}$
- (b)  $\sqrt{5}$
- (c)  $\frac{15}{\sqrt{5}}$
- (d)  $\frac{75}{\sqrt{5}}$

**Hints:**

AP series where c.d. is  $\frac{1}{\sqrt{5}}$  and hence the answer is (d)

# Time and Distance

## 1.4

### Introduction:

Time and Distance have a mathematical relationship with Speed. There exists a dependence of speed, time and distance on each other.

For example,

In case, the distance is specified in km, the time taken to cover would be in hrs then the unit of speed would be given in km/hr. While, on the other hand, if the distance is specified in m and the time taken to cover is given in sec, the unit of speed would be m/sec.

### Relation between Time and Distance with Speed

1. **Distance = Speed × Time**

Unit of Distance is denoted as km or m.

2.  **$Time = \frac{Distance}{Speed}$**

Unit of Time is denoted as hr, min or sec.

3.  **$Speed = \frac{Distance}{Time}$**

Unit of Speed is denoted as km/hr or m/sec.

**Exercise:****Theoretical Questions**⊙ **Multiple Choice Questions (MCQ)****[Ratio, Variation and Proportion]**

1. If  $Q:W = 2:3$ ,  $W:T = 4:5$  and  $T:Y = 6:7$ , find  $Q:Y$ .
  - (a) 35: 51
  - (b) 16: 35
  - (c) 16: 51
  - (d) 35: 51
2. Two numbers are in the ratio 7: 9, if the sum of the numbers is 288, then the smaller number is
  - (a) 126
  - (b) 288
  - (c) 162
  - (d) 144
3. When the number 1580 is increased in the ratio 5: 9, the new number is
  - (a) 788
  - (b) 2488
  - (c) 878
  - (d) 2844
4. The ratio of the number of faces to the number of edges of a box is
  - (a) 3: 8
  - (b) 8: 3
  - (c) 1: 2
  - (d) 2: 1
5. Find the fourth proportional to 6, 8, 9 is
  - (a) 18
  - (b) 12
  - (c) 7
  - (d) 13
6. Find the compounded ratio of: 15: 8 and 18: 5
  - (a) 27: 4
  - (b) 4: 27
  - (c) 27: 8
  - (d) 8: 27

7. Find the third proportional to  $6\frac{1}{4}$  and 5
- (a)  $9\frac{1}{2}$
  - (b)  $3\frac{1}{5}$
  - (c) 4
  - (d) 8
8. Find the Duplicate ratio of :  $8a : \sqrt{4b}$
- (a)  $64a^2 : 2b^2$
  - (b)  $16a^2 : b^2$
  - (c)  $32a^2 : 4b^2$
  - (d)  $32a^2 : 8b^2$
9. If  $A \propto \frac{1}{B}$  and  $A = 7$  when  $B = 3$ , then when  $B = 2\frac{1}{3}$ ,  $A$  is
- (a)  $\frac{3}{7}$
  - (b) 3
  - (c) 9
  - (d) 1
10. If  $AB \propto XY$  and  $\frac{A}{B} + \frac{B}{A} = \frac{X}{Y} + \frac{Y}{X}$  then
- (a)  $A^2 + B^2 \propto X^2 + Y^2$
  - (b)  $A + B \propto XY$
  - (c)  $AB \propto X + Y$
  - (d)  $A + B \propto \frac{1}{X + Y}$
11. Using the properties of proportion, solve the following equation for  $y$ , given  $\frac{341}{91} = \frac{y^3 + 3y}{3y^2 + 1}$
- (a) 14
  - (b) 11
  - (c) 12
  - (d) 10

12. The ratio of 45 seconds to  $5\frac{3}{4}$  minutes is
- (a) 5: 21
  - (b) 3:23
  - (c) 2: 19
  - (d) 1: 27
13. A class consists of 48 male students and 23 female students. Find the ratio of female students to total strength of the class.
- (a) 23: 48
  - (b) 48: 23
  - (c) 48: 71
  - (d) 23: 71
14. The mean proportional between 80 and  $\frac{1}{5}$  is
- (a) 16
  - (b) 4
  - (c) 8
  - (d) 32
15. Find the Triplicate Ratio of:  $\frac{1}{3}:\frac{1}{2}$
- (a) 8: 27
  - (b) 27: 8
  - (c) 4: 9
  - (d) 9: 4
16. If 4, 6,  $p$ , 27,  $q$  are in continued proportion, find the values of  $p$  and  $q$ .
- (a)  $p = 9, q = 9$
  - (b)  $p = 9, q = 81$
  - (c)  $p = 81, q = 9$
  - (d)  $p = 81, q = 81$
17. If a Stadium having spectators divided into groups of Young Generation and Old Generation and the number of young spectators are 1525 and old spectators are 1875. Find the ratio of Total Spectators to Young Spectators.
- (a) 136:75
  - (b) 136:85
  - (c) 75:61
  - (d) 136:61

18. Find two numbers whose mean proportional is 8 and the 1<sup>st</sup> number is square of the 2<sup>nd</sup> number.
- (a) 8, 8
  - (b) 4, 2
  - (c) 16, 4
  - (d) 64, 8
19. What number must be subtracted from each of the numbers 12, 13, 24 and 29 to make them proportional?
- (a) 9
  - (b) 8
  - (c) 35
  - (d) 36
20. The ratio of the pocket money saved by Rakesh and his sister is 5:6. If the sister saves ₹60 more, how much more the brother should save in order to keep the ratio of their savings unchanged?
- (a) ₹60
  - (b) ₹40
  - (c) ₹50
  - (d) ₹70

**[Time value of Money and Annuity – Simple and Compound Interest]**

21. AMS & Co. borrows ₹1,20,000 for one year at 15% annual interest, compounded monthly. Find their monthly payment.
- (a) ₹11,380
  - (b) ₹10,830
  - (c) ₹11,430
  - (d) ₹10,740
22. A certain sum of money invested at a certain rate of compound interest doubles in 8 years. In how many years will it become 16 times?
- (a) 31 years
  - (b) 28 years
  - (c) 30 years
  - (d) 32 years
23. Calculate the present value of an annuity of ₹3,000 received at the beginning of each year for 5 years at a discount factor of 6%
- (a) ₹13,998
  - (b) ₹13,398

- (c) ₹14,398  
(d) ₹14,898
24. AB LLP is expecting to receive a certain sum of money five years from now. If the present value of this sum is ₹38,400 at current market interest rate of 6% when the interest is compounded monthly, then how much amount they will receive after 5 years?
- (a) ₹52,684  
(b) ₹52,884  
(c) ₹51,904  
(d) ₹51,794
25. What sum will amount to ₹5480 in 6 years at 10% p.a. compound interest payable half-yearly?
- (a) ₹3,051  
(b) ₹2,051  
(c) ₹3,501  
(d) ₹2,501
26. Mr. Singh deposits ₹2,000 at the beginning of each year for 5 years. How much do these accumulate at the end of 5th year at an interest rate of 6%?
- (a) ₹10,272  
(b) ₹11,591  
(c) ₹10,727  
(d) ₹11,951
27. A buys a shop for ₹48,750 down payment and ₹60,000 after one year. If the money is worth 10% per year compounded half-yearly. Find the purchase price of the shop.
- (a) ₹94,280  
(b) ₹1,03,170  
(c) ₹45,530  
(d) ₹54,420
28. A loan of ₹20,000 has been issued for 5 years. Compute the amount to be repaid to the lender if simple interest is charged @ 8% per year.
- (a) ₹28,000  
(b) ₹25,000  
(c) ₹27,000  
(d) ₹24,000

29. A sum of ₹4895 borrowed from a money lender at 5% p.a. compounded annually. Find the annual instalment, if amount borrowed is to be paid back in three equal annual instalments.
- ₹ 1,597
  - ₹ 1,997
  - ₹ 1,397
  - ₹ 1,797
30. AMS & Co. makes a monthly payment for ₹11,350 for one year at 11% annual interest, compounded monthly. Find the amount borrowed by them.
- 1,31,889
  - 1,38,284
  - 1,22,025
  - 1,22,802
31. Calculate the present value of an annuity of ₹5,000 received annually for 4 years at a discount rate of 5%.
- ₹ 17,720
  - ₹ 12,250
  - ₹ 22,250
  - ₹ 27,720
32. Find the Principal, if the compound value is ₹9,750 at the interest rate of 6.25% p.a. compounded annually for 16 years.
- ₹ 4,287
  - ₹ 3,857
  - ₹ 3,696
  - ₹ 4,857
33. Find the amount and the compound interest of ₹9,350 at the rate of 8% p.a. compounded half-yearly for four years.
- ₹ 12,795 and ₹ 3,445
  - ₹ 12,720 and ₹ 3,370
  - ₹ 12,758 and ₹ 3,408
  - ₹ 12,835 and ₹ 3,485
34. When the effective rate of interest is 7.82% payable quarterly, what would be the nominal rate of interest?
- 8.00%
  - 7.60%
  - 7.00%
  - 8.20%

35. Find the present value of perpetuity of ₹2,725 at 0.50% per month compound interest.
- (a) ₹45,167
  - (b) ₹45,417
  - (c) ₹44,517
  - (d) ₹44,157
36. A sum of money double itself at 7% p.a. compound interest in
- (a) 10.27 years
  - (b) 10.00 years
  - (c) 12.07 years
  - (d) 12.00 years
37. A Laptop depreciated in value each year at 15% of its previous value and at the end of fourth year, the value was ₹36,331. Find its original value.
- (a) ₹69,600
  - (b) ₹69,200
  - (c) ₹69,300
  - (d) ₹69,500
38. A wants to create an endowment fund of ₹51,750 to provide for a prize of every year. If the fund can be invested at 11% p.a. compound interest, find the amount of prize.
- (a) ₹5,933
  - (b) ₹5,963
  - (c) ₹5,993
  - (d) ₹5,693
39. Accumulated series of deposits as future sum money is classified as –
- (a) Annuity Fund
  - (b) Sinking Fund
  - (c) Marginal Fund
  - (d) Nominal Fund
40. The compound interest on ₹27,500 at 8% p.a. is ₹7,142. Calculate the required time.
- (a) 3 years
  - (b) 2.5 years
  - (c) 3.5 years
  - (d) 4 years

**[Arithmetic Progression and Geometric Progression]**

**Solve Q. Nos. 41 to 50 using the following information.**

**For the series 5, 12, 19, 26, 33, 40 .....**

41. Identify the type of given series.
  - (a) A.P.
  - (b) G.P.
  - (c) H.P.
  - (d) None of the Above
42. Find the value of the 12th term of the series.
  - (a) 89
  - (b) 92
  - (c) 82
  - (d) 75
43. Which term of the series would be 68?
  - (a) 10
  - (b) 9
  - (c) 12
  - (d) 8
44. What would be the value of the 9th term?
  - (a) 66
  - (b) 68
  - (c) 59
  - (d) 61
45. What would be the sum of the first 11 terms of the series?
  - (a) 419
  - (b) 440
  - (c) 420
  - (d) 426
46. Which term of the series would be 82?
  - (a) 12
  - (b) 11
  - (c) 10
  - (d) 14

47. If the sum of the series is 297, how many terms are there in the series?
- (a) 8
  - (b) 9
  - (c) 10
  - (d) 7
48. What is the value of 'a'?
- (a) 40
  - (b) 12
  - (c) 5
  - (d) 7
49. Find the value of 'd'?
- (a) 7
  - (b) 5
  - (c) -7
  - (d) -5
50. Value of 'd' is \_\_\_\_?
- (a) Zero
  - (b) Negative
  - (c) Positive
  - (d) Undefined
51. Find the next 5 terms for the series: 2, -4, 8, -16, 32
- (a) -64, 128, -256, 512, -1024
  - (b) 64, -128, 256, -512, 1024
  - (c) 64, 128, 256, 512, 1024
  - (d) -64, -128, -256, -512, -1024
52. Find the value of 'r' for the series: 729, 243, 81, 27, and 9
- (a) 3
  - (b) -3
  - (c) 1/9
  - (d) 1/3
53. For any series having 50 terms forming A.P. with first term equal to 25, what will be the value of 'n'?
- (a) 50
  - (b) 25

- (c) 75  
 (d) 100
54. Find the next 6 terms for the series: 128, 139, 150, 161, 172, 183.  
 (a) 194, 205, 216, 228, 240, 252  
 (b) 194, 205, 217, 229, 242, 256  
 (c) 194, 205, 218, 231, 245, 259  
 (d) 194, 205, 216, 227, 238, 249
55. For the given series: 66, 71, 76, 81, 86, 91, 96 ....666, 671, 676.  
 With 'n' terms in the series, what will be the value of 'd' ?  
 (a) -5  
 (b) 15  
 (c) -3  
 (d) 5
56. Identify the type of series and the value of 'a': -85, -83, -81, -79, -77  
 (a) A.P. , a = -85  
 (b) A.P. , a = -2  
 (c) G.P. , a = -2  
 (d) G.P. , a = -85
57. For any G.P. starting with 28, if each term is obtained by dividing the preceding one by 2, what would be value of 'r' ?  
 (a) 2  
 (b) -2  
 (c) 1/2  
 (d) -1/2
58. If the First term is 748, Last Term is 28, and the value between two consecutive terms is 9 deducted, find the number of terms in the series.  
 (a) 9  
 (b) 8  
 (c) 10  
 (d) None of the above
59. If the Last term is 187, First Term is 371, value between two consecutive terms is 8 deducted, find the number of terms in the series.  
 (a) 25  
 (b) 22

(c) 23

(d) 24

60. Identify the type of series: 1, 2, 3, 4, 5

(a) H.P.

(b) G.P.

(c) Either of 'a' or 'b'

(d) A.P.

**[Time and Distance]**

61. When a Bike had travelled for 78 km in 3 hours 45 minutes in the evening, how much distance would the Bike travelled in 2 hours?

(a) 41.60 km

(b) 48.88 km

(c) 52 km

(d) 55 km

62. A cycle travels a distance of 300 m in every second. What is the distance covered in an hour by the cycle?

(a) 3000 km

(b) 1080 km

(c) 1800 km

(d) 2160 km

63. A Car takes 4 hours to travel 90 km. If time is reduced by 15 minutes, how much distance would be covered in the given time?

(a) 84.375 km

(b) 80 km

(c) 95.625 km

(d) 90 km

64. A Train takes 35 hours to reach Punjab from Kolkata (1940 km) and takes 42 hours to reach Kolkata from Gujarat (2160 km). But it took 25 hours from Punjab to Gujarat (1420 km). How many days it take for a trip from Kolkata-Punjab-Gujarat-Kolkata and what is the distance covered?

(a) 4 days 6 hours and 5520 km

(b) 3 days 18 hours and 5520 km

(c) 4 days and 2680 km

(d) 5 days and 2680 km

65. If Bike had travelled at the speed of 40km/hr for 2 hours and scooty had travelled at the speed of 25km/hr for 3 hours, which one of them covered more distance and how much more was covered?
- (a) Scooty, 30km
  - (b) Bike, 80 km
  - (c) Bike, 5 km
  - (d) Scooty, 75km
66. If Bike covers 165 km in 3 hours and scooty covers 100 km in 2 hours, in order to cover 1025 km, how much time would be required by bike and scooty?
- (a) 12.42 hours and 12.42 hours
  - (b) 21.50 hours and 19.63 hours
  - (c) 19.52 hours and 19.52 hours
  - (d) 18.63 hours and 20.50 hours
67.  $y$  is the yardstick to measure the performance of two vehicles, where  $y = \text{Speed} \times \text{Time} \times \text{Distance}$ . If Time taken by one of the vehicle (1st Vehicle) is increased by 10%, what would be the impact on the yardstick?
- (a) No change
  - (b) 1st vehicle would be better than 2nd Vehicle
  - (c) 2nd Vehicle would be better than 1st Vehicle
  - (d) None of the Above
68. If a student is travelling at a speed of 45 km/hr, how long should it take to travel 90 km?
- (a) 2 hours
  - (b) 3 hours
  - (c) 100 minutes
  - (d) 90 minutes
69. With 5% increase in Distance and 2.50% increase in Time, what would be the impact on Speed?
- (a) Decrease by 2.44%
  - (b) Increase by 2.44%
  - (c) Increase by 2.38%
  - (d) Decrease by 2.38%
70. If a Car covered 200 km at a speed of 25kmph and a Bike covered 150 km at a speed of 30kmph, which one of them took more time to cover the distance and what was the time taken?
- (a) Car, 3hrs
  - (b) Bike, 8 hrs
  - (c) Car, 8 hrs
  - (d) Bike, 5 hrs

71. If Toto had travelled at the speed of 15km/hr for 11 hours and Car had travelled at the speed of 45km/hr for 3.5 hours, which one of them covered shorter distance and how much short ?
- (a) Car, 7.50 km
  - (b) Toto, 7.50 km
  - (c) Car, 15 km
  - (d) Toto, 15 km
72. If ratio between Speed(km/hr.) and Time(Hrs) is 7: 5, what will be the distance(km)?
- (a) 30 km
  - (b) 35 km
  - (c) 36 km
  - (d) 42 km
73. The distance between two terminal stations of Metro is 240 km. A metro rail takes 2 hours to cover the distance. Calculate the total distance covered in 5 days, if 10 trips to and fro takes place between the two stations in a day?
- (a) 6,000 km
  - (b) 12,000 km
  - (c) 24,000 km
  - (d) 18,000 km
74. If A takes 3 hours to cover a distance of 60 km, B takes 2 hours to cover a distance of 50 km, how much time would A take more than B to cover 300 km?
- (a) 6 hours
  - (b) 12 hours
  - (c) 15 hours
  - (d) 3 hours
75. If Toto covered 240 km at a speed of 40kmph while Scooty covered 180 km at a speed of 60 kmph, which one of them took less time to cover the distance and what was the time taken ?
- (a) Scooty, 3 hrs.
  - (b) Toto, 6 hrs
  - (c) Toto, 3 hrs.
  - (d) Scooty, 6 hrs.

76. With 10% decrease in Distance and 5% increase in Time, what would be the impact on Speed?
- Decrease by 16.67%
  - Increase by 16.67%
  - Increase by 14.28%
  - Decrease by 14.28%
77. The distance between two terminal metro stations A & B is 300 km. A metro rail takes 1.50 hours to cover the distance. Calculate the total time worked by the driver (in terms of days) in a week, if 6 trips from Station A to Station B takes place in a day?
- 2.625 days
  - 5.25 days
  - 2.50 days
  - 4.25 days
78.  $y$  is the yardstick to measure the performance of two vehicles, where  $y = \text{Speed} \times \text{Time} \times \text{Distance}$ . If Distance travelled by one of the vehicle (2nd Vehicle) is increased by 2%, what would be the impact on the yardstick?
- 1st Vehicle would travel 4% more distance
  - 2nd Vehicle would travel 4% more distance
  - No change in the distance travelled by any of the vehicle
  - None of the above
79. If a Toto travels for 6 hours on a working day to cover 1.5 times the distance travelled on holiday and on a holiday it takes 4 hours to cover 110 km. than for a span of 7 days: (5 Working and 2 Holidays), how much distance is covered?
- 1054 km
  - 1045 km
  - 880 km
  - 990 km
80. If ratio between Time (seconds) and Speed (m/s) is 12:13, what will be the distance (m)?
- 156 m
  - 1.56 km
  - 0.0156 m
  - 15.6 km

⊙ **State True or False**

1. Mean Proportional of  $p$  and  $q$  is same as their Mean.
2. If  $3x + 4y : 5x - 3y = 5 : 3$  then  $x : y = 27 : 16$
3. The ratio of two numbers is 12:5. If antecedent is 45 then the consequent is 108
4. If the ratio of two positive numbers is 4:5 and their L.C.M is 140 then the number are 35, 45
5. The compound ratio of sub- duplicate ratio and sub-triplicate ratio of 729:64 is 81:8
6. The ratio of two numbers is 11:15. The sum of 3 times the first number and twice the second numbers is 630. The H.C.F of the numbers is 10
7. The mean proportional of  $4x$  and  $16x^3$  is  $12x^2$
8. The third proportional of 1 hr 20 minutes, 1hr 40 minutes is 2 hrs.
9. The fourth proportional of ₹5, ₹3.50, and 150 gm. is 125 gm.
10. If  $A : B = B : C = C : D = 5 : 6$  then  $A : B : C : D = 125 : 150 : 180 : 216$
11. If the first and third numbers of four positive number is continued proportion be 3 and 12 respectively then fourth number is 36
12. A sum of money amounts to ₹720 in 2 years and ₹789 in 3 years. The rate of simple interest is 12%
13. The S.I at  $x\%$  for  $x$  years will be ₹ $X$  on a sum of  $x$
14. The S.I on a sum of money at 8% p.a for 6years is half the sum. The sum is ₹640
15. If the interest on ₹1,200 more than the interest on ₹1,000 by ₹50 in 3 years then the rate of Interest is  $8\frac{1}{3}\%$
16. The rate of S.I p.a when a sum of money grows to one and a half times itself in 8 years is  $6\frac{1}{4}\%$
17. The C.I on a certain sum of money for 1year at 8% p.a compounded quarterly is ₹824 then the sum is ₹10,033
18. In 3 years ₹1,600 amount to ₹1,936 at 10% p.a C.I
19. The C.I on ₹5,000 for 2 years at a certain rate of interest p.a amounts to ₹1,050 then the rate of interest is 10%
20. The difference between S.I and C.I on ₹1,000 for 1 year at 4% payable quarterly is Re. 0.40

● **Fill in the Blanks**

1. ₹2530 is distributed between Ram and Hari such that Ram gets  $\frac{11}{12}$  part of that Hari gets. Then Hari gets \_\_\_\_\_
2. Some amount of money is distributed among Rama, Mitra and Shipra such that twice the money that Rama gets = thrice the amount of money that Mitra gets = four times the amount of money that Shipra gets. Then the continued ratio of their money is \_\_\_\_\_
3. In a map 2cm denotes a distance of 3 km., then the scale in the map is \_\_\_\_\_
4. The ratio of two numbers is 2:3. If 6 is subtracted from the second number then the number which is subtracted from the first number so that the new ratio becomes the same as that of the previous, is \_\_\_\_\_
5. The sub-duplicate ratio of 49:81 is \_\_\_\_\_
6.  $\left(\frac{1}{2} + \frac{1}{3}\right) : \left(\frac{1}{2} \times \frac{1}{3}\right) : \text{_____}$
7. The compound ratio of 1.2:2.5, 2.1:3.2 and 5:3 is: \_\_\_\_\_
8. If A: B = 3:4, B: C = 2:5, then A: B: C: \_\_\_\_\_
9. Two numbers are in the ratio is 5:8 and if 6 be subtracted from each of them then the reminders are in the ratio 1:2, then the numbers are \_\_\_\_\_
10. If 3, x, 27 are in continued proportion then x = \_\_\_\_\_
11. For the ratio 8:27, the Triplicate Ratio of its Subtriplicate Ratio is \_\_\_\_\_
12. A certain sum of money at S.I amounts to ₹500 in 3 years and to ₹600 in 5 years then the principal is \_\_\_\_\_
13. In \_\_\_\_\_ time the Simple interest on ₹900 at 6% will be equal to S.I on ₹540 for 8 years at 5%
14. Due to fall in the rate of simple interest from 12% to  $10\frac{1}{2}\%$  p.a money lender's yearly income diminished by ₹90. The capital is \_\_\_\_\_
15. A sum was put at S.I at a certain rate for 2 years. Had it been put at 2% higher rate, it would have fetched ₹100 more. The sum is \_\_\_\_\_
16. At \_\_\_\_\_ percent rate, the Simple interest will be equal to the principal amount in 10 years.
17. I lent ₹4,000 for 9 months at 12% p.a. If C.I is reckoned quarterly what will I get after 9 months is \_\_\_\_\_
18. The C.I on ₹5,000 for 3 years, when the rate of interest is 5%, 6% and 7% for the 1st, 2nd and 3rd years respectively, is \_\_\_\_\_
19. The difference between S.I and C.I of ₹4,000 for  $2\frac{1}{2}$  years @ 10% p.a is \_\_\_\_\_
20. The sum of money will amount to ₹6,050 in 2 years at 10% p.a. C.I is \_\_\_\_\_
21. The C.I on a certain sum of money for 2 years at 8% p.a. compounded annually is ₹1,040. The sum is \_\_\_\_\_

**Answer:**

⊙ **Multiple Choice Questions (MCQ)**

Q. No.	Answer	Remarks
1	(b)	$Q : W = 2:3$ , $W : T = 4:5$ and $T : Y = 6:7$ $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7}$ $\frac{48}{105}$ $\frac{16}{35}$ or $16:35$
2	(a)	Ratio = $7:9$ Sum of numbers = 288 Sum of ratios = $7+9 = 16$ Smaller number = $288/16 \times 7 = 126$
3	(d)	$1580$ increased in $5 : 9$ $1580/5 \times 9 = 2844$
4	(c)	No. of faces : 6 No. of edges : 12 Ratio of no. of faces to edges = $6 : 12 = 1 : 2$
5	(b)	Let Fourth proportional to 6, 8, 9 be $x$ $\frac{6}{8} = \frac{9}{x}$ or, $x = \frac{9 \times 8}{6} = \frac{72}{6} = 12$
6	(a)	$15 : 8$ and $18 : 5$ $\frac{15}{8} \times \frac{18}{5}$ $\frac{3}{4} \times 9$ $\frac{27}{4}$ or $27:4$
7	(c)	Third proportional to $6\frac{1}{4}$ and 5 is $6\frac{1}{4} : 5 :: 5 : Y$ $\frac{25}{4} : 5 :: 5 : Y$ $\frac{25}{4} \times Y = 5 \times 5$ $\frac{Y}{4} = 1$ $Y = 4$

8	(b)	$8a : \sqrt{4b}$ is the given ratio $(8a)^2 : (\sqrt{4b})^2 =$ Duplicate Ratio of the given one $= 64a^2 : 4b^2$ $= 16a^2 : b^2$
9	(c)	$A \propto \frac{1}{B} \therefore A = \frac{K}{B}$ Given $A = 7$ when $B = 3 \therefore 7 = \frac{K}{3}$ or, $K = 21$ So when $B = 2\frac{1}{3} = \frac{7}{3}$ then $A = \frac{21}{7/3}$ or, $A = 9$
10	(a)	$AB \propto XY \therefore AB = K.XY$ $\frac{A}{B} + \frac{B}{A} = \frac{X}{Y} + \frac{Y}{X}$ Or, $\frac{A^2 + B^2}{AB} = \frac{X^2 + Y^2}{XY}$ Or, $\frac{A^2 + B^2}{K.XY} = \frac{X^2 + Y^2}{XY}$ Or, $A^2 + B^2 = K(X^2 + Y^2)$ $\therefore A^2 + B^2 \propto X^2 + Y^2$
11	(b)	$\frac{341}{91} = \frac{y^3 + 3y}{3y^2 + 1}$ or, $\frac{341 + 91}{341 - 91} = \frac{(y^3 + 3y) + (3y^2 + 1)}{(y^3 + 3y) - (3y^2 + 1)}$ [By Componendo and Dividendo] $\frac{432}{250} = \frac{(y^3 + 3y + 3y^2 + 1)}{(y^3 + 3y - 3y^2 - 1)}$ or, $\frac{216}{125} = \frac{(y + 1)^3}{(y - 1)^3}$ or, $\frac{(y + 1)^3}{(y - 1)^3} = \frac{6^3}{5^3}$ or, $\frac{(y + 1)}{(y - 1)} = \frac{6}{5}$ or, $5 \times (y + 1) = 6 \times (y - 1)$ or, $5y + 5 = 6y - 6$ or, $6y - 5y = 5 + 6$ or, $Y = 11$

12	(b)	$45 \text{ Seconds} : 5\frac{3}{4} \text{ Minutes} = \frac{45 \text{ Seconds}}{(5\frac{3}{4} \times 60) \text{ Seconds}} = \frac{45 \text{ Seconds}}{345 \text{ Seconds}}$ $= \frac{3}{23}$
13	(d)	<p>Male Students – 48 , Female students – 23,  Total strength of the class : Male students + Female students  Total strength : (48+23) = 71 students.  Ratio of female students to Total strength = 23 : 71</p>
14	(b)	<p>Mean Proportional between 80 and <math>\frac{1}{5}</math> is</p> $\sqrt{80 \times \frac{1}{5}}$ $= \sqrt{16} = 4$
15	(a)	<p><math>\frac{1}{3} : \frac{1}{2} =</math> Given Ratio</p> <p><math>\left(\frac{1}{3}\right)^3 : \left(\frac{1}{2}\right)^3 =</math> Triplicate Ratio of the given ratio</p> $= \frac{1}{27} : \frac{1}{8}$ $= \frac{1}{27} \times \frac{8}{1}$ $= \frac{8}{27} \text{ or } 8:27$
16	(b)	<p>4, 6, <math>p</math>, 27, <math>q</math> are in continued proportion</p> $4/6 = 6/p \text{ and } p/27 = 27/q$ <p>or, <math>4 \times p = 6 \times 6</math></p> <p>or, <math>4 \times p = 36</math></p> <p>or, <math>p = 36/4 = 9</math></p> $p/27 = 27/q$ <p>or, <math>p \times q = 27 \times 27</math></p> <p>or, <math>9 \times q = 729</math></p> <p>or, <math>q = 729/9 = 81</math></p> <p><math>p = 9</math> and <math>q = 81</math></p>
17	(d)	<p>Young Spectators – 1525 , Old Spectators – 1875,  Total strength of the stadium = Young Spectators + Old Spectators  Total strength = (1525+1875) = 3400 spectators.  Ratio of Total Strength to Young Spectators = 3400 : 1525 = 136 : 61</p>

18	(c)	<p>Mean proportional is 8  1<sup>st</sup> number is square of 2<sup>nd</sup> number ,  Let two numbers be <math>x^2</math> and <math>x</math>,  Mean proportional of the numbers = <math>\sqrt{x \times x^2}</math>  <math>8 = \sqrt{x^3}</math> (Given)  or, <math>8^2 = (\sqrt{x^3})^2</math>  or, <math>64 = x^3</math>  or, <math>4^3 = x^3</math>  or, <math>x = 4</math>  or, <math>x^2 = 16</math>  Two numbers are 16 and 4.</p>
19	(a)	<p>Number must be subtracted from 12, 13, 24 and 29 to make them proportional be <math>y</math>,  <math>(12-y) : (13-y) :: (24-y) : (29-y)</math>  <math>(12-y) \times (29-y) = (24-y) \times (13-y)</math>  or, <math>348 - 12y - 29y + y^2 = 312 - 24y - 13y + y^2</math>  or, <math>348 - 41y + y^2 = 312 - 37y + y^2</math>  or, <math>41y - 37y = 348 - 312</math>  or, <math>4y = 36</math> or, <math>y = 9</math></p>
20	(c)	<p>Let the savings of Rakesh and his sister be <math>5x</math> and <math>6x</math>,  And Rakesh would save ₹ <math>y</math> more,  <math>\frac{(5x+y)}{(6x+60)} = \frac{5}{6}</math>  or, <math>(5x+y) \times 6 = 5 \times (6x+60)</math>  or, <math>30x+6y = 30x+300</math>  or, <math>6y = 300</math>  or, <math>y = \frac{300}{6} = 50</math></p>

21	(b)	$V = 120000, r \% = 15 \text{ and monthly interest rate } (r) = \frac{0.15}{12} = 0.0125$ $\text{Monthly payment } (P) = \frac{V \times r}{1 - (1+r)^{-n}}$ $P = \frac{120000 \times 0.0125}{1 - (1 + 0.0125)^{-12}}$ $P = \frac{1500}{1 - 0.8615}$ $P = \frac{1500}{0.1385}$ $P = 10830.32 \text{ or } 10,830 \text{ (Approx.)}$
22	(d)	<p>If amount doubles in 8 years,</p> $2P = P(1+r)^8$ $2 = (1+r)^8$ <p>Amount would be 16 times,</p> $16P = P(1+r)^n$ $16 = (1+r)^n$ $2^4 = (1+r)^n$ <p>Replacing 2 with <math>(1+r)^8</math>, we get</p> $(1+r)^{8 \times 4} = (1+r)^n$ $(1+r)^{32} = (1+r)^n$ $n = 32 \text{ years}$
23	(b)	$P = ₹3,000, r = \frac{6}{100} = 0.06, n = 5$ $V = \frac{P \times (1+r)}{r} \times \left[ 1 - \frac{1}{(1+r)^n} \right]$ $V = \frac{3000 \times (1+0.06)}{0.06} \times \left[ 1 - \frac{1}{(1+0.06)^5} \right]$ $V = 53000 \times \left[ 1 - \frac{1}{1.3382} \right]$ $V = 53000 \times [1 - 0.7472]$ $V = 53000 \times 0.2528 = 13,398.40. \text{ or } 13,398 \text{ (Approx.)}$

24	(d)	$P = ₹38,400, r = \frac{6}{100} = 0.06, n = 5$ $P = \frac{A}{\left(1 + \frac{r}{12}\right)^{12 \times n}}$ $38400 = \frac{A}{\left(1 + \frac{0.06}{12}\right)^{12 \times 5}}$ $38400 = \frac{A}{(1 + 0.005)^{60}}$ $38400 = \frac{A}{1.3488}$ $A = 38400 \times 1.3488 = 51,793.92 \text{ or } 51,794 \text{ (Approx.)}$
25	(a)	$A = 5480, n = 6, m = 2, r = \frac{10}{100} \text{ or } 0.10$ $A = P \times \left(1 + \frac{r}{m}\right)^{mn}$ $5480 = P \times \left(1 + \frac{0.10}{2}\right)^{2 \times 6}$ $5480 = P \times (1.05)^{12}$ $5480 = P \times 1.7959$ $P = \frac{5480}{1.7959} = 3051.39 \text{ or } 3,051 \text{ (Approx.)}$
26	(d)	$P = 2000, r = \frac{10}{100} \text{ or } 0.06, n = 5$ $A = \frac{P \times (1+r)}{r} [(1+r)^n - 1]$ $A = \frac{2000 \times (1+0.06)}{0.06} [(1+0.06)^5 - 1]$ $A = \frac{2120}{0.06} [1.3382 - 1]$ $A = 35333.33 \times 0.3382$ $A = 11949.73 \text{ or } 11950 \text{ (Approx.)}$
27	(b)	$A \text{ (Present Value)} = 60000, r = \frac{10}{100} \text{ or } 0.10$ $P = A \left(1 + \frac{r}{2}\right)^{-2n}$ $P = 60,000 \times \left(1 + \frac{0.10}{2}\right)^{-2 \times 1}$ $P = 60,000 \times (1.05)^{-2}$ $P = 60,000 \times 0.9070 = 54,420$ $\text{Purchase price of shop} = 48750 + 54420 = 103170$

28	(a)	$P = 20,000, n = 5 \text{ years}, r = 8\% \text{ or } 0.08$ $A = P + P \times n \times r$ $A = 20,000 + 20,000 \times 5 \times 0.08$ $A = 20,000 + 8,000$ $A = 28,000$
29	(d)	$V = 4895, r = 0.05, n = 3$ $V = \frac{P}{r} \times [1 - (1+r)^{-n}]$ $4895 = \frac{P}{0.05} \times [1 - (1+0.05)^{-3}]$ $4895 \times 0.05 = P \times [1 - (1.05)^{-3}]$ $244.75 = P \times [1 - 0.8638]$ $244.75 = P \times 0.1362$ $P = \frac{244.75}{0.1362} = 1796.99 \text{ or } 1,797 \text{ (Approx.)}$
30	(c)	$A = 11,350 \times 12 = 1,35,660, r\% = 11 \text{ and monthly interest rate} = \frac{0.11}{12} = 0.0092, n = 1 \text{ year}$ $A = P \times \left(1 + \frac{r}{12}\right)^{12n}$ $136200 = P (1 + 0.0092)^{12}$ $P = 1,22,025.39 = 1,22,025 \text{ (approx.)}$
31	(a)	$A = 5,000, r = \frac{5}{100} = 0.05, n = 4$ $P = \frac{A}{r} \left[1 - \frac{1}{(1+r)^n}\right]$ $P = \frac{5,000}{0.05} \left[1 - \frac{1}{(1+0.05)^4}\right]$ $P = 100000 \left[1 - \frac{1}{1.2155}\right]$ $P = 100000 [1 - 0.8227]$ $P = 100000 \times 0.1772$ $P = 17720$
32	(c)	$A = 9750, r = 6.25\% \text{ p.a. or } 0.0625, n = 16$ $A = P (1+r)^n$ $\text{Or, } 9750 = P (1 + 0.0625)^{16}$ $\text{Or, } 9750 = P \times 2.63793$ $\text{Or, } P = 3696.08$

33	(a)	$P = ₹9,350, r = \frac{8}{100} = 0.08, n = 4$ $A = P\left(1 + \frac{r}{2}\right)^{2n}$ $A = P\left(1 + \frac{0.08}{2}\right)^{2 \times 4}$ $A = 9350(1 + 0.04)^{2 \times 4}$ $A = 9350(1.04)^8$ $A = 9350 \times 1.3685$ $A = 12795$ $CI = A - P = 12795 - 9350 = 3445$
34	(b)	$0.0782 = \left(1 + \frac{r}{4}\right)^{1 \times 4} - 1$ $0.0782 = \left(1 + \frac{r}{4}\right)^4 - 1$ $0.0782 + 1 = \left(1 + \frac{r}{4}\right)^4$ $\left(1 + \frac{r}{4}\right)^4 = 1.0782$ <p>Taking logarithms on both sides, we get</p> $4 \times \log\left(1 + \frac{r}{4}\right) = \log 1.0782$ $\log\left(1 + \frac{r}{4}\right) = \frac{0.0327}{4}$ $\log\left(1 + \frac{r}{4}\right) = \log 0.008175$ <p>Anti-logarithms on both sides, we get</p> $\left(1 + \frac{r}{4}\right) = 1.0190$ $\frac{r}{4} = 1.0190 - 1$ $\frac{r}{4} = 0.0190$ $r = 0.0190 \times 4 = 7.60\%$
35	(b)	$P = 2,725, r = 0.50\% \text{ per month} = 0.50 \times 12 = 0.06 = 6\% \text{ p.a.}$ $V = \frac{P}{r}$ $V = \frac{2725}{0.06} = 45416.66 \text{ or } 45,417 \text{ (Approx.)}$

36	(a)	$2P = P(1 + 0.07)^n$ $2 = (1.07)^n$ $\log 2 = \log (1.07)^n$ $0.3010 = n \times 0.0293$ $n = \frac{0.3010}{0.0293} = 10.27 \text{ years}$
37	(a)	$A = ₹36,331, r = \frac{15}{100} = 0.15, n = 4$ $A = P(1 - r)^n$ $36331 = P(1 - 0.15)^4$ $36331 = P(0.85)^4$ $36331 = P \times 0.5220$ $P = \frac{36331}{0.5220} = 69599.62 \text{ or } 69,600 \text{ (Approx.)}$
38	(d)	$V = 51750, r = \frac{11}{100} \text{ or } 0.11$ $V = \frac{P}{r}$ $51750 = \frac{P}{0.11}$ $P = 51750 \times 0.11 = 5,692.50 \text{ or } 5,693 \text{ (Approx.)}$
39	(b)	Accumulated series of deposits as future sum money is classified as -Sinking Fund
40	(a)	$A = 27500 + 7142 = 34642, P = 27500, r = 8\% \text{ p.a.} = 0.08$ $A = P \times (1 + r)^n$ $34642 = 27500 \times (1 + 0.08)^n$ $\frac{34642}{27500} = (1 + 0.08)^n$ $1.2597 = (1.08)^n$ $(1.08)^3 = (1.08)^n$ $n = 3 \text{ years.}$
41	(a)	Type of Series – A.P.
42	(c)	$\text{Value of the } 12^{\text{th}} \text{ the term} = a + (n - 1) \times d = 5 + (12 - 1) \times 7$ $= 5 + 11 \times 7 = 5 + 77 = 82$
43	(a)	<p>Term of the series would be 68</p> $\text{Value} = a + (n - 1) \times d$ $68 = 5 + (n - 1) \times 7 \text{ or, } 68 - 5 = (n - 1) \times 7 \text{ or, } 63 = (n - 1) \times 7$ $\text{or, } 63/7 = (n - 1) \text{ or, } 9 = n - 1 \text{ or, } n = 9 + 1 = 10$

44	(d)	Value of 9 <sup>th</sup> term = $a + (n-1) \times d = 5 + (9-1) \times 7$ $= 5 + 8 \times 7 = 5 + 56 = 61$
45	(b)	Sum of first 11 terms of the series $n/2 \times \{2a + (n-1) \times d\} = (11/2) \times \{2 \times 5 + (11-1) \times 7\}$ $= (11/2) \times \{10 + 10 \times 7\} = (11/2) \times \{10 + 70\}$ $= (11/2) \times \{80\} = 11/2 \times 80 = 11 \times 40 = 440$
46	(a)	term of the series would be 82 Value = $a + (n-1) \times d$ $82 = 5 + (n-1) \times 7$ or, $82 - 5 = (n-1) \times 7$ or, $77 = (n-1) \times 7$ or, $77/7 = (n-1)$ or, $11 = n-1$ or, $n = 11+1 = 12$
47	(b)	sum of the series is 297 $n/2 \times \{2a + (n-1) \times d\} = 297$ or, $n \times \{2 \times 5 + (n-1) \times 7\} = 297 \times 2$ or, $n \times \{10 + 7n - 7\} = 594$ or, $n(7n + 3) = 594$ or, $7n^2 + 3n - 594 = 0$ or, $7n^2 + 66n - 63n - 594 = 0$ or, $n(7n + 66) - 9(7n + 66) = 0$ or, $(7n + 66) = 0$ & $(n-9) = 0$ or, $n = -66/7$ & $n = 9$ Since, $n$ cannot be negative, there are 9 terms ( $n = 9$ ).
48	(c)	Value of 'a' = 1 <sup>st</sup> term of the Series = 5
49	(a)	Value of 'd' = 2 <sup>nd</sup> Term - 1 <sup>st</sup> Term = $12 - 5 = 7$
50	(c)	Value of 'd' = 2 <sup>nd</sup> Term - 1 <sup>st</sup> Term = $12 - 5 = 7$ Since $7 > 0$ , the value of 'd' is Positive
51	(a)	Next 5 terms for the series : 2, -4, 8, -16, 32 Since, type of Series is G.P. 'r' = 2 <sup>nd</sup> term / 1 <sup>st</sup> term = $-4/2 = -2$ Next 5 terms would be $[32 \times (-2)]$ , $[32 \times (-2)^2]$ , $[32 \times (-2)^3]$ , $[32 \times (-2)^4]$ , $[32 \times (-2)^5]$ -64, 128, -256, 512, -1024
52	(d)	value of 'r' for the series : 729, 243, 81, 27, 9 $r = 2^{\text{nd}} \text{ Term} / 1^{\text{st}} \text{ Term} = 243/729 = 1/3$
53	(a)	The series is having 50 terms, Therefore, value of 'n' would be 50

54	(d)	<p>Next 6 terms for the series :</p> <p>128, 139, 150, 161, 172, 183</p> <p>Since, type of Series – A.P.</p> <p>'<math>d</math>' = 2<sup>nd</sup> term – 1<sup>st</sup> term = 139 – 128 = 11</p> <p>Next 6 terms would be</p> <p>(183+11), (183+11+11), (183+11+11+11), (183+11+11+11+11), (183+11+11+11+11+11), (183+11+11+11+11+11)</p> <p>194, 205, 216, 227, 238, 249</p>
55	(d)	<p>Series : 66, 71, 76, 81, 86, 91, 96, .....666, 671, 676</p> <p>Type of Series – A.P.</p> <p>'<math>d</math>' = 2<sup>nd</sup> term – 1<sup>st</sup> term = 71 – 66 = 5</p>
56	(a)	<p>Type of series and the Value of '<math>a</math>' :</p> <p>–85, –83, –81, –79, –77</p> <p>Type of series : A.P.</p> <p>'<math>a</math>' : First Term = –85</p>
57	(c)	<p>G.P. series which is starting with 28, each term divided by 2, what would be value of '<math>r</math>'</p> <p>Since the Common Ratio is 1/2, value of '<math>r</math>' = 1/2</p>
58	(d)	<p>If the First term is 748, Last Term is 28, value between two consecutive terms is 9 deducted</p> $t_n = a + (n-1) \times d$ <p>or, 28 = 748 + (n-1) × (-9)</p> <p>or, 28 – 748 = (n-1) × (-9)</p> <p>or, -720 = (n-1) × (-9)</p> <p>or, -720 / (-9) = n-1</p> <p>or, 80 = n-1</p> <p>or, n = 80 + 1 = 81</p>
59	(d)	<p>If the Last term is 187, First Term is 371, value between two consecutive terms is obtained by deducting 8, find the number of terms in the series</p> $t_n = a + (n-1) \times d$ <p>or, 187 = 371 + (n-1) × (-8)</p> <p>or, 187 – 371 = (n-1) × (-8)</p> <p>or, -184 = (n-1) × (-8)</p> <p>or, -184 / (-8) = n-1</p> <p>or, 23 = n-1</p> <p>or, n = 23+1 = 24</p>

60	(d)	series : $1+2+3+4+5$ Type of Series – A.P.
61	(a)	In the Evening, Bike covered 78 km in 3 hours 45 minutes, 45 minutes in an hour = $45/60 = 0.75$ Total time the bike travelled in the evening = $3+0.75 = 3.75$ hrs If 78 km covered in 3.75 hrs, then in 1 hr = $78/3.75 = 20.80$ km If travelled for 2 hours then it would cover = $20.80 \times 2 = 41.60$ km
62	(b)	Distance = 300 m Time taken = 1 second for every 300 m Total time = 1 hour = 60 minutes = $60 \times 60$ seconds = 3600 seconds Total Distance to be covered in an hour = $300 \text{ m} \times 3600 \text{ seconds} = 1080000 \text{ m}$ We know, 1000 m = 1 km So, 1080000 m = 1080 km
63	(a)	Ideal Situation : Distance – 90 km in 4 hours When time is reduced by 15 minutes ( $15/60 = 0.25$ hr) Time available = $4 - 0.25 = 3.75$ hrs Distance can be covered = $90/4 \times 3.75 = 84.375$ km
64	(a)	Punjab from Kolkata (1940 km) – 35 hours Kolkata from Gujarat (2160 km)- 42 hours Punjab to Gujarat (1420 km)-25 hours Trip from Kolkata-Punjab-Gujarat-Kolkata: Total Distance = $1940 + 2160 + 1420 = 5520$ km Total Time taken (in Hrs.) = $35 + 42 + 25 = 102$ hours = $102/24$ (As 1 Day = 24 hours) = 4.25 days or 4 days and 6 hours
65	(c)	Bike : Speed – 40km/hr and Time -2 hrs Distance (km) = Speed $\times$ Time = $40 \times 2 = 80$ km Scooty : Speed – 25km/hr and Time -3 hrs Distance (km) = Speed $\times$ Time = $25 \times 3 = 75$ km Therefore, Bike travelled 5 km more than Scooty (80–75 km).

66	(d)	<p>Bike covers 165 km in 3 hours : In an hour = <math>165/3 = 55</math> km</p> <p>Scooty covers 100 km in 2 hours : In an hour = <math>100/2 = 50</math> km</p> <p>In order to cover 1025 km,</p> <p>Bike = <math>1025/55 = 18.63</math> hours</p> <p>Scooty = <math>1025/50 = 20.50</math> hours</p>
67	(a)	<p>Yard stick (y) = Speed <math>\times</math> Time <math>\times</math> Distance</p> <p>Where, speed = Distance / Time</p> <p>So, <math>y = \text{Distance} / \text{Time} \times \text{Time} \times \text{Distance}</math></p> <p><math>y = (\text{Distance})^2</math></p> <p>Therefore, any change in Time, yardstick would not have any impact. So, no change in the yardstick due to increase in time.</p>
68	(a)	<p>Travelling Speed - 45 kmph</p> <p>Distance to be covered - 90 km</p> <p>Speed (kmph) = Distance (km) / Time(hours)</p> <p>Time = Distance / Speed</p> <p>Time (hours) = <math>90 / 45 = 2</math> hours</p>
69	(b)	<p>Let Distance be 100 – Increase by 5% - New Distance = 105</p> <p>Time be 50 – Increase by 2.5% - New Time = 51.25</p> <p>Speed = <math>100/50 = 2</math>,</p> <p>New Speed = <math>105/51.25 = 2.0488</math></p> <p>Change in Speed = <math>2.0488 - 2 = 0.0488</math></p> <p>Increase (%) = <math>0.0488/2 \times 100 = 2.44\%</math></p>
70	(c)	<p>For Car :</p> <p>Distance - 200 km, Speed – 25 kmph</p> <p>Time = Distance / Speed = <math>200/25 = 8</math> hrs</p> <p>For Bike :</p> <p>Distance – 150 km, Speed – 30 kmph</p> <p>Time = Distance / Speed = <math>150/30 = 5</math> hrs</p> <p>Car took <math>(8 - 5) = 3</math> hours more than Bike.</p>
71	(a)	<p>Toto :</p> <p>Speed – 15km/hr and Time-11 hrs</p> <p>Distance (km) = Speed <math>\times</math> Time = <math>15 \times 11 = 165</math> km</p> <p>Car :</p> <p>Speed – 45km/hr and Time-3.50 hrs</p> <p>Distance (km) = Speed <math>\times</math> Time = <math>45 \times 3.50 = 157.50</math> km</p> <p>Therefore, Car travelled 7.50 km less than Toto (165-157.50 km).</p>

72	(b)	Speed 7 km/hr, Time – 5hrs $\text{Speed (km/hr) = Distance (km) / Time(hr)}$ $\text{Distance (km) = Speed (km/hr) } \times \text{ Time(hr)}$ $= 7 \times 5 = 35 \text{ km}$
73	(c)	Distance between two terminal stations of Metro is 240 km. Total distance covered in 5 days, if 10 trips to and fro takes place between the two terminal stations $\text{Distance in single trip} = 240 \times 2 = 480 \text{ km}$ $\text{In a Day, 10 trips (to and fro)} = 480 \times 10 = 4800 \text{ km}$ $\text{For 5 days, } 4800 \text{ km} \times 5 = 24,000 \text{ km}$
74	(d)	A takes 3 hours to cover a distance of 60 km, $\text{To cover 300 km, A would take} = 3/60 \times 300 = 15 \text{ hours}$ B takes 2 hours to cover a distance of 50 km, $\text{To cover 300 km, B would take} = 2/50 \times 300 = 12 \text{ hours}$ A would take 3 hours more than B (15–12 hours)
75	(a)	For Toto : $\text{Distance - 240 km, Speed – 40 kmph}$ $\text{Time} = \text{Distance} / \text{Speed} = 240/40 = 6 \text{ hrs}$ For Scooty : $\text{Distance – 180 km, Speed – 60 kmph}$ $\text{Time} = \text{Distance} / \text{Speed} = 180/60 = 3 \text{ hrs}$ Scooty took $6 - 3 = 3$ hours less time than Toto.
76	(d)	Let Distance be 100 – Decrease by 10% - New Distance = 90 $\text{Time be 50 – Increase by 5% - New Time} = 52.50$ $\text{Speed} = 100/50 = 2,$ $\text{New Speed} = 90/52.50 = 1.7143$ $\text{Change in Speed} = 1.7143 - 2 = -0.2857$ $\text{Decrease (\%)} = -0.2857/2 \times 100 = 14.28\%$
77	(a)	Distance between two terminal metro stations is 300 km. Total distance covered in a week (7 days), 6 trips from A to B takes place $\text{In a Day, 6 trips} = 300 \times 6 = 1800 \text{ km}$ $\text{For 7 days, } 1800 \text{ km} \times 7 = 12,600 \text{ km}$ $\text{Time taken for a week} = 1.5/300 \times 12600 = 63 \text{ hours} = 2.625 \text{ days}$

78	(b)	<p>Yardstick (y) = Speed × Time × Distance  Where, speed = Distance / Time  So, y = Distance / Time × Time × Distance  <math>y = (\text{Distance})^2</math>  Therefore, with Increase in Distance of 2<sup>nd</sup> Vehicle by 2%, the distance would be increased by 4%.</p>
79	(b)	<p>Toto travels for 6 hours on a working day to cover 1.5 times the distance travelled on holiday  On a holiday it takes 4 hours to cover 110 km  Holiday – 110 km in 4 hours  Working day – <math>110 \times 1.5 = 165</math> km in 6 hours  For a span of 7 days : 5 Working and 2 Holidays  Distance = <math>5 \times 165 + 2 \times 110 = 825 + 220 = 1045</math> km</p>
80	(a)	<p>Let Time(s) 12 seconds, Speed (m/s) – 13 m/s  Speed (m/s) = Distance(m) / Time(s)  Distance (m) = Speed(m/s) × Time(s)  = <math>12 \times 13 = 156</math> m</p>

⊙ **State True or False**

1	2	3	4	5	6	7	8	9	10
F	T	F	F	F	T	F	F	F	T
11	12	13	14	15	16	17	18	19	20
F	T	F	F	T	T	T	F	T	F

⊙ **Fill in the Blanks**

1	₹1,320	2	6:4:3
3	1:150000	4	4
5	7:9	6	5:1
7	21:40	8	3:4:10
9	15, 24	10	±9
11	8:27	12	₹350
13	4 year	14	₹6,000
15	₹2,500	16	10%
17	₹4,370.91	18	₹954.55
19	₹76.23	20	₹1,050
21	₹6,250		

# ALGEBRA

# 2

## **This Module includes**

- 2.1 Set Theory, Venn Diagram**
- 2.2 Indices and Logarithms (Basic Concepts)**
- 2.3 Permutation and Combinations (Basic Concepts)**
- 2.4 Quadratic Equations (Basic Concepts)**

# ALGEBRA

## **Module Learning Objectives:**

After studying this Module, the students will be able to understand –

- ✦ The concepts of Set alongwith it's representation by Venn diagram
- ✦ What is meant by Indices and it's relationship with Logarithm
- ✦ The basic concepts of Permutation and Combination
- ✦ The theories of Quadratic Equations

# Set Theory, Venn Diagram

## 2.1

### Definition of Set

Any collection of things or objects can be defined as set when the things or objects of the collection are distinctly different and well-defined. By ‘well-defined’, we understand that all the things or objects of that collection must belong to the collection.

Each object or thing in the collection is termed as Element of the Set.

For example,

On a table, 5 journals, 5 magazines and 1 laptop are placed. In this scenario, 5 journals and 5 magazines can be part of the “Books and Periodicals” Collection while the laptop cannot be part of such collection. So, 5 journals and 5 magazines are be part of the Set “Books and Periodicals” while the Laptop is not a part of the Set.

### Methods for Representation of Sets

#### 1. Tabular or Roster Method

When the elements can be identified and are explicitly given, the method of representation called is Tabular or Roster Method. In this method, the elements are separated by comma and are written one after another. Such Sets are denoted within brackets [ ] or braces { }.

For example,

The set of odd numbers between: 11 to 30.

Let A be the set of odd numbers between 11 to 30,

$$A = \{11, 13, 15, 17, 19, 21, 23, 25, 27, 29\}$$

The set of even numbers between: 1 to 25.

Let N be the set of even numbers between 1 to 25,

$$N = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24\}$$

#### 2. Set-Builder Method or Property Method

Most general method of representing set is Property or Set-Builder Method. In this method, all the elements of the set are not listed or mentioned rather the common properties shared by all the elements are stated in the Set.

For example,

If  $W$  be the set of all even numbers between: 81 to 100. For selecting any arbitrary number within the range as  $t$ , we write as follows:

$$W = \{t: t \text{ is even and } 81 < t < 100\}$$

Understanding the expression:

$W$  is a set of numbers  $t$  where  $t$  is an even number and lies between: 81 to 100.

### Types of Set

#### 1. Set of Sets

When all the elements of a set are sets in themselves, that set is called set of sets.

For Example,

$$T = \{a, b, c, d\}$$

$$\text{Where, } a = \{1, 2, 3, 4\}; b = \{1, 3, 5, 7\}; c = \{2, 4, 6, 8\}; d = \{1, 3, 4, 6\}$$

Here,  $T$  would be termed as Set of Sets.

#### 2. Singleton Set

If any set has only one element, it is termed as Singleton Set. In simple terms, set with just one element is called unit set or singleton set.

For example,

$$\text{Given: Set } L = \{33\}$$

Meaning: Set  $L$  has only one element which is 33.

#### 3. Null Set

Void Set or Empty Set or Null Set is a set which do not have any element and is denoted by  $\{\}$  or  $\emptyset$ .

It may be noted that the Null Set  $\emptyset$  is distinctly different from the Set  $\{0\}$ . The former contains no element at all, whereas the latter contains only one element i.e. 0.

#### 4. Finite Set

If the number of elements can be counted for a given set, that set is termed as Finite Set. Finite Set comprises of specific number of elements.

#### 5. Infinite Set

When the number of elements to be counted in a given set is infinite, the set is termed as Infinite Set. So, Infinite set does not have specific number of elements.

#### 6. Equal Set

For any two given sets, if each of the elements of one set are same as each of the elements of the other set, then the sets are called Equal Set.

For Example,

If all elements of set  $Y$  are same to all the elements of set  $W$ ,  $Y = W$  can be written to signify that every element of Set  $Y$  is equal to every element of Set  $W$ .

## 7. Subset

For any two given sets, if one set has all the elements of the other set, the other set is termed as subset of the first one.

For example,

Let  $A = \{2,4,5,6,7,8,9\}$  and  $B = \{2,4,6,8\}$

Since, B has all the elements present in A, Set B is called a subset of Set A.

Denoted as  $B \subseteq A$  or  $A \supseteq B$

Here A is called a Superset of B.

Important points to note:

- (i) Every Set is a Subset of itself
- (ii) Empty Set is a Subset of every set i.e.  $\emptyset \subseteq A$ .
- (iii) Every Set has at least two Subsets – the Empty Set and the Set itself.

## 8. Proper Subset and Superset

For any given two sets, if one set has all the elements of the other set and also at least one more element is present in it, the other set is termed as proper subset of the first one. While, the first one is termed as Superset of the other one.

For example,

Let  $A = \{2,4,5,6,7,8,9\}$  and  $B = \{2,4,6,8\}$

Since, A has all the elements present in B and some other elements as well, Set B is called a proper subset of Set A. While, Set A is termed as Superset of Set B.

Denoted as  $B \subset A$  and  $A \supset B$ .

Important points to note :

- (i) Set A is not a Proper Subset of itself
- (ii) A is a Proper Subset of B,  $A \subset B$  if all the elements of Set A are also the elements of Set B as well as  $A \neq B$

## 9. Equivalent sets [0]

Two finite sets  $A$  and  $B$  are said to be equivalent if there is one to one correspondence between the elements of  $A$  and  $B$  and we write  $A \equiv B$ .

From the definition it can be said that two finite sets  $A$  &  $B$  are equivalent if and only if they have same number of elements.

i.e.  $n(A) = n(B)$ .

The symbol '0' is used to denote equivalence of sets

Let  $A = \{a, e, i, o, u\}$  then  $n(A) = 5$

And  $B = \{1, 2, 3, 4, 5\}$  then  $n(B) = 5$

So  $A \equiv B$  as  $n(A) = n(B)$

## 10. Universal Set

Universe of Discourse or Universal Set is a fixed set. If all the sets which are under consideration are subsets of a Fixed Set then the fixed set is called as Universal Set. The Universal Set is also called Space and its elements are called Points. The symbol  $U$  or  $S$  is used to denote the Universal Set.

For Example,

$$Q = \{-10, -9, -8, -7, \dots, -1\}; W = \{-100, -80, -60, -40, \dots, 40, 60\}$$

$$E = \{15, 30, 45, 60, 75, 90\}; R = \{-90, -60, -30, \dots, 60, 90\}$$

$$S = \{-100, -99, \dots, 99, 100\}$$

The elements of each of the sets ( $Q, W, E, R$ ) are present in the Set  $S$ , that means, all the sets ( $Q, W, E, R$ ) are subsets of  $S$ . Hence,  $S$  is the Universal Set.

## 11. Power Set

Set comprises of all the subsets of a given set, is termed as Power Set of the given set. It is denoted by  $P(V)$ , considering set  $V$  is the given set. Let  $V = \{0, 1, 2\}$ .  $P(V) = [\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}]$

The number of elements of Power set can be determined as follows:

For any finite set  $M$  with  $n$  elements, the power set of  $M$  would have  $2^n$  elements. In the above example, given set  $V$  has 3 elements. So no of elements to of  $P(V)$  is  $2^3 = 8$ .

## 12. Product Set

The Product Set of two given sets  $A$  &  $B$  is the set whose elements are the Ordered Pairs  $(a, b)$  where 'a' is an element of set  $A$  &  $B$  'b' is an element of Set  $B$ . It is written as  $A \times B$  & read as "A cross B". It can be noted that  $A \times B$  &  $B \times A$  are different.

Let  $A = \{1,2\}$  &  $B = \{a,b,c\}$  then  $A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$  and  $B \times A = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

**Cardinality (or) order of a set**

The no. of distinct elements in a set is called the cardinality (or) order of the set. If a finite set  $A$  has  $n$  distinct elements, then the cardinality of the set  $A$  is  $n$  and is denoted by  $0(A)$  or  $n(A)$

$$\text{If } A = \{a, b, c\} \text{ then } n(A) = 3$$

$$B = \{1, 2, 3\} \text{ then } 0(B) = 3$$

$$\therefore n(A) = 0(B)$$

**Note:** The order of empty set is zero.

**Important points to be kept in mind**

- Every Set would be a Subset of its' own Set.  
Denoted as  $Q \subseteq Q$ .
- Null Set would be a Subset of every Set.  
Denoted as  $\emptyset \subseteq Q$ .
- When there are 3 sets –  $Q, W$  and  $R$  and  $Q \subseteq W$  and  $W \subseteq R$ , then  $Q \subseteq R$ .

In simple terms,

If Q is a Subset of W and W is a Subset of R then Q is a Subset of R.

### Set Operations

#### 1. Joint or Union

When two sets are joined, it is termed as Union of Sets. In simple terms, the union of two different sets  $A$  &  $B$  results in a new set with all the elements of the  $A$  &  $B$  appearing only once. It is denoted as  $A \cup B$ .

Let  $A = \{1,3,5,7\}$  and  $B = \{5,7,9\}$  So  $A \cup B = \{1,3,5,7,9\}$ . This operation is possible for more than two sets also.

#### 2. Meet or Intersection

When a set is formed with the common elements of two or more sets, the new set is said to be the result of Intersection of sets. If  $A$  &  $B$  are the two given sets then their Intersection is denoted as  $A \cap B$ . When  $A = \{2,3,4\}$  and  $B = \{3,4,6\}$  then  $A \cap B = \{3,4\}$

#### 3. Disjoint

When two sets do not have any elements in common, both the sets are termed as disjoint to each other.

Two sets  $A$  and  $B$  are said to be disjoint sets if they have no element in common i.e.,  $A \cap B = \emptyset$ .

If  $A = \{1, 3, 5\}$ ,  $B = \{2, 4\}$

Then  $A \cap B = \emptyset$ . Hence  $A$  &  $B$  are disjoint sets

#### 4. Difference

Of the two sets, a new set formed by the elements of the first which do not form part of the other is, the end result of the operation Difference of sets.

In other words, the Difference of two sets  $A$  &  $B$  is the set of all elements of  $A$  which do not belong to  $B$ . It is written as  $A - B$ .

Let  $A = \{5,7,8,9\}$  and  $B = \{9\}$  then  $A - B = \{5,7,8\}$

#### 5. Complement or Negation

When  $A$  is the sub-set of the universal set  $U$ , the elements of Complement of  $A$  would comprise of all the elements of  $U$  except the elements of  $A$ . It is denoted as  $A'$  or  $A^c$  or  $\bar{A}$ .

If  $A = \{1,2,3,4\}$  and  $U = \{1,2,3,4,5,6,7\}$  then  $A' = U - A = \{5,6,7\}$ .

### Laws of Algebra of Sets

#### 1. Idempotent :

(i)  $A \cap A = A$

(ii)  $A \cup A = A$

While considering two or more sets, the term idempotent means the sets would be same and identical even after the operation.

2. Commutative :

(i)  $A \cap B = B \cap A$

(ii)  $A \cup B = B \cup A$

While solving two or more sets, the commutative law states that the elements of both the sets would be added.

3. Associative :

(i)  $A \cup (B \cap C) = (A \cup B) \cap C$

(ii)  $A \cap (B \cup C) = (A \cap B) \cup C$

When two or more sets are to be combined, the order of the sets would not matter since the result would be same if the sets are associative to each other.

4. Distributive :

(i)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

When two or more sets are to be distributed, the order of the sets would not matter since the result would be same if the sets are distributive.

5. De Morgan's Law :

(i)  $(A \cup B)' = A' \cap B'$

Complement of the Union of two sets is the Intersection of the Complement of both the sets.

(ii)  $(A \cap B)' = A' \cup B'$

Complement of the Intersection of two sets is the Union of the Complement of both the sets.

6. Identity Law:

(i)  $A \cup U = U$

(ii)  $A \cup \phi = A$

(iii)  $A \cap \phi = \phi$

(iv)  $A \cap U = A$

For any set A in relation to Universal Set U and Null Set, the above mentioned laws are applicable.

7. Complement Law :

(i)  $A \cup A' = U$

(ii)  $\Phi' = U$

(iii)  $U' = \Phi$

(iv)  $A \cap A' = \Phi$

(v)  $(A')' = A$

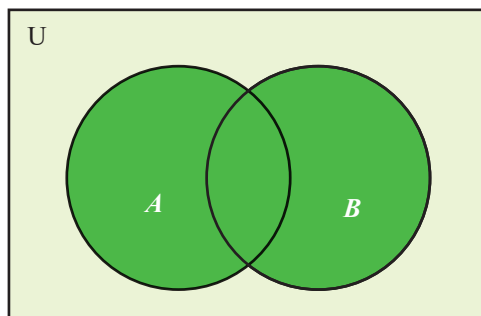
For any set A in relation to Universal Set U and Null Set, the above mentioned laws are applicable.

## VENN DIAGRAM

Pictorial representation of the relationship between two or more sets, is known as Venn diagram. It was introduced by Euler and Developed by John Venn. In this diagram, sets are represented in closed curves by points. Each sets are represented by Circles and form a part of Rectangle (which is representing the Universal Set).

### Set operation using Venn diagram

1. Joint or Union ( $A \cup B$ )

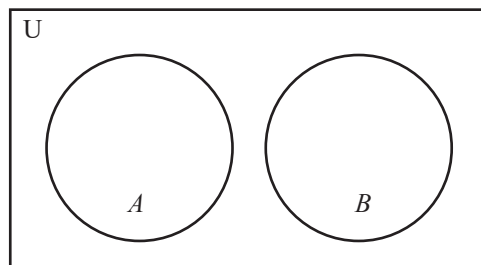


Either A or B or both

$$A \cup B = \blacksquare$$

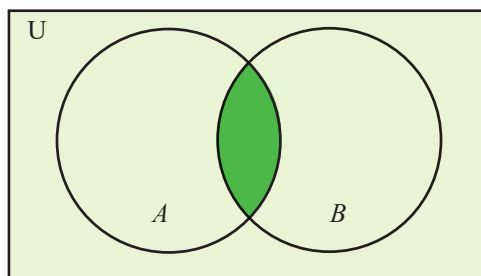
**Fig. 1:** Venn Diagram representing  $A$  union  $B$

2. Disjoint



**Fig. 2:** Venn Diagram showing  $A$  and  $B$  are disjoint sets

3. Meet or Intersection ( $A \cap B$ )

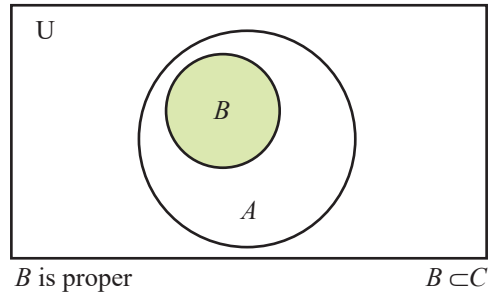


Both A and B

$$A \cap B = \blacksquare$$

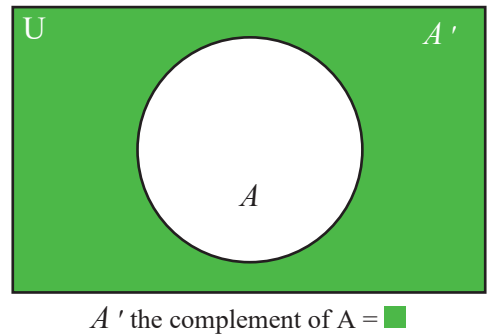
**Fig. 3:** Venn Diagram representing  $A$  intersection  $B$

4. Proper Subset



**Fig. 4:** Venn Diagram representing  $B$  is proper subset of  $A$

5. Complement or Negation ( $A'$ )



**Fig. 5:** Venn Diagram representing complement of Set  $A$

**Important results related to Set Operations**

1.  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2.  $n(A \cup B) = n(A) + n(B)$  when  $A$  &  $B$  are disjoint sets
3.  $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
4.  $n(A \cup B \cup C) = n(A) + n(B) + n(C)$  when  $A, B$  &  $C$  are disjoint sets

**Illustration 1.**

Write the set-builder form of set for:  $M = \{1, 8, 27, 64, 125 \dots\dots\dots\}$

**Solution:**

The elements of the set  $M$  are cube of all natural numbers.

Hence,  $M$  in set-builder form would be expressed as follows:

$$M = \{x: x = n^3, n \in \mathbb{N}\}$$

**Illustration 2.**

Write the roster form of set for:  $B = \{x: x \text{ is a letter in the word 'LANGUAGE'}\}$

**Solution:**

The word 'LANGUAGE' has 8 letters, while the distinct letters are: L, A, N, G, U, E

$$B = \{L, A, N, G, U, E\}$$

**Illustration 3.**

The Set with their elements are mentioned below :

$$A = \{\text{All numbers}\}$$

$$B = \{2, 4, 6, 8, 10\}$$

$$C = \{1, 3, 5, 7, 9\}$$

$$D = \{ \}$$

Identify the different types of sets.

**Solution:**

A - Universal Set - It comprises of All Numbers.

B and C - Sub-sets of A that is Sub-sets of U.

D – Null Set – It comprises of no elements.

**Illustration 4.**

If Set A = { 2, 4, 6, 8, 10 }; Set B = { 1, 3, 5, 7, 9 }; Set C = { 1, 2, 3, 4, 5 } and Set D = { 6,7,8,9,10 }.

Find:

(i)  $A \cup B$

(ii)  $A \cap C$

(iii)  $B \cap D$

(iv)  $A \cap D$

**Solution:**

(i)  $A \cup B = \{ 1,2,3,4,5,6,7,8,9,10 \}$

(ii)  $A \cap C = \{2, 4\}$

(iii)  $B \cap D = \{7, 9\}$

(iv)  $A \cap D = \{6, 8, 10\}$

**Illustration 5.**

If Q = {1, 2, 3, 5} and W = {3, 4, 5, 7}. Find the Union of Q and W.

**Solution:**

Q - Contains 4 elements: 1,2,3,5

W – Contains 4 elements: 3,4,5,7

$Q \cup W$  – Contains 6 elements since no elements should repeat in case of union of sets.

$$Q \cup W = \{1, 2, 3, 4, 5, 7\}$$

**Illustration 6.**

If Q = {-1, -2, 8, 10} and W = {6, 9, -2, -1}. Find the Intersection of Q and W.

**Solution:**

Q - Contains 4 elements: -1, -2, 8, 10

W – Contains 4 elements: 6, 9, -2, -1

$$Q \cap W = \{-1, -2\}$$

**Illustration 7.**

If  $U = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$  and  $A = \{0, 1, 2, -1, -2\}$ , find  $A'$ .

**Solution:**

All the elements which are part of  $U$  but not part of  $A$ , would be part of  $A'$ .

$$A' = \{-5, -4, -3, 3, 4, 5\}$$

**Illustration 8.**

If  $T = \{q, w, e, r, t, y\}$  and  $Y = \{w, r, y, u, p, o\}$ , find the difference  $(Y-T)$ .

**Solution:**

Set  $(Y-T)$  – All the elements of Set  $Y$  which are not part of Set  $T$ .

$$\text{Set } (Y-T) = \{u, p, o\}.$$

**Illustration 9.**

If  $U = \{-2, 1, 2, 3, 4, 5\}$  and  $A = \{-2, 5\}$ , find the elements of set  $(U - A')$ .

**Solution:**

We know that,  $U - A = A'$ , hence,  $U - A' = A$ .

$$\text{So, set } (U - A') = \text{set } A = \{-2, 5\}$$

**Illustration 10.**

Rewrite the following examples using set notation

- (i) First ten even natural numbers.
- (ii) Set of days of a week.
- (iii) Set of months in a year which have 30 days.
- (iv) The numbers 3, 6, 9, 12, 15
- (v) The letters m, a, t, h, e, m, a, t, I, c, s

**Solution:**

- (i)  $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$   
 $= \{x: x \text{ is an even integer and } 2 \leq x \leq 20\}$
- (ii)  $A = \{\text{Sunday, Monday, } \dots, \text{Saturday}\}$   
 $= \{x: x \text{ is a day in a week}\}$
- (iii)  $A = \{\text{April, June, September, November}\}$   
 $= \{x: x \text{ is a month of a year which has 30 days}\}$

(iv)  $A = \{x: x \text{ is a positive number multiple of 3 and } 3 \leq x \leq 15\}$

(v)  $A = \{x: x \text{ is a letter in the word mathematics}\}$

### Illustration 11.

Write the following set in roster form.

(i)  $A = \{x: x \text{ is an integer, } -3 < x < 7\}$

(ii)  $B = \{x: x \text{ is an integer, } 4 < x < 12\}$

#### Solution:

(i)  $A = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$

(ii)  $B = \{6, 8, 10, 12\}$

### Illustration 12.

Represent the following sets in a selector method:

(i) All numbers less than 15

(ii) All even numbers

#### Solution:

Taking  $R$  to be the set of all real numbers in every case:

(i)  $\{x: x \in R \text{ and } x < 15\}$

(ii)  $\{x: x \in R \text{ and } x \text{ is a multiple of } 2\}$

### Illustration 13.

$A = \{1, 2, 3, 4, 6, 7, 12, 17, 21, 35, 52, 56\}$ ,  $B$  and  $C$  are subsets of  $A$  such that  $B = \{\text{odd numbers}\}$ ,  $C = \{\text{prime numbers}\}$ . List the elements of the set  $\{x: x \in B \cap C\}$

#### Solution:

$B \cap C = \{1, 3, 7, 17, 21, 35\} \cap \{2, 3, 7, 17\} = \{3, 7, 17\}$

Therefore, reqd. list =  $\{3, 7, 17\}$

### Illustration 14.

If  $A = \{1, 2, 3, 4, 5\}$ ,  $B = \{2, 4, 5, 8\}$ ,  $C = \{3, 4, 5, 6, 7\}$ , find  $A \cup (B \cap C)$ .

#### Solution:

$B \cap C = \{2, 3, 4, 5, 6, 7, 8\}$ ,  $A \cup (B \cap C) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

### Illustration 15.

If  $A = \{1, 2, 3\}$  and  $B = \{2, 3, 4\}$ ; find  $(A - B) \cup (B - A)$

#### Solution:

$A - B = \{1\}$ ,  $B - A = \{4\}$ ,  $(A - B) \cup (B - A) = \{1, 4\}$

**Illustration 16.**

If  $S$  is the set of all prime numbers,  $M = \{x : x \text{ is an integer, } 0 \leq x \leq 9\}$  exhibit

- (i)  $M - (S \cap M)$   
 (ii)  $M \cup N$ , where  $N = \{0, 1, 2, \dots, 20\}$

**Solution:**

$S = \{2, 3, 5, 7, 11, 13, \dots\}$ ,  $M = \{0, 1, 2, \dots, 8, 9\}$

- (i)  $S \cap M = \{2, 3, 5, 7\}$ ,  $M - (S \cap M) = \{0, 1, 4, 6, 8, 9\}$   
 (ii)  $M \cup N = \{0, 1, \dots, 20\}$

**Illustration 17.**

In a class of 100 students, 45 students read Physics, 52 students read Chemistry and 15 students read both the subjects. Find the number of students who study neither Physics nor Chemistry.

**Solution:**

We know  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ . Let  $A$  indicates the set of students who read Physics and  $B$  for Chemistry.

Given:

$$n(A) = 45, n(B) = 52, n(A \cap B) = 15$$

$$\text{So, } n(A \cup B) = 45 + 52 - 15 = 82$$

$$\text{We are to find } n(A' \cap B') = 100 - n(A \cup B) = 100 - 82 = 18$$

**Illustration 18.**

In a survey of 1000 families it is found that 454 use electricity, 502 use gas, 448 use kerosene, 158 use gas and electricity, 160 use gas and kerosene and 134 use electricity and kerosene for cooking. If all of them use at least one of the three, find how many use all the three fuels.

**Solution:**

Let us take  $E$  for set of families use electricity,  $G$  for gas, and  $K$  for kerosene.

$$\text{Now } n(E) = 454, n(G) = 502, n(K) = 448$$

$$n(G \cap E) = 158, n(G \cap K) = 160, n(E \cap K) = 134, n(E \cap G \cap K) = ?$$

$$n(E \cup G \cup K) = 1000$$

$$\text{Again } n(E \cup G \cup K) = n(E) + n(G) + n(K) - n(E \cap G) - n(G \cap K) - n(K \cap E) + n(E \cap G \cap K)$$

$$\text{Or, } 1000 = 454 + 502 + 448 - 158 - 160 - 134 + n(E \cap G \cap K) = 952 + n(E \cap G \cap K)$$

$$\text{Or, } n(E \cap G \cap K) = 1000 - 952 = 48.$$

**Illustration 19.**

If  $A = \{\text{letters of word ASSASSINATION}\}$

$B = \{\text{letters of the word POSSESSION}\}$

Find  $A \cap B$

**Solution:**

$$A \cap B = \{S, S, S, S, I, O, N\}$$

**Illustration 20.**

Find no. of subjects formed from the letters of the word 'ALLAHABAD'.

**Solution:**

$$2^5 = 3^2 \quad (A, L, H, B, D)$$

**Illustration 21.**

$$\text{If } A = \{p, q, r, s\}, B = \{q, s, t\}, C = \{m, q, n\}$$

Find  $C - (A \cap B)$

**Solution:**

$$C - (A \cap B)$$

$$= (mqn) - (q, s) = (m, n)$$

**Illustration 22.**

$$\text{If } n(A) = 24, n(B) = 22 \text{ \& } n(AB) = 8$$

Find  $n(B - A)$

**Solution:**

$$n(B - A) = n(B) - n(AB)$$

$$= 22 - 8$$

$$= 14$$

**Illustration 23.**

If  $A = (0, 2)$ , Find  $n\{P(A)\}$ .

**Solution:**

Power set of  $A = \{(0, 2), \emptyset, (0)(2)\}$

$$n\{P(A)\} = 4$$

**Illustration 24.**

If  $n(A - B) = 24$ ,  $n(B - A) = 19$  and  $n(AB) = 11$ , find a)  $n(A)$  b)  $n(B)$  c)  $n(A \cup B)$ .

**Solution:**

$$\text{a) } n(A) = n(A - B) + n(AB) = 24 + 11 = 35$$

$$\text{b) } n(B) = n(B - A) + n(AB) = 19 + 11 = 30$$

$$\text{c) } n(A \cup B) = n(A - B) + n(B - A) + n(AB)$$

$$= 24 + 19 + 11 = 54$$

**Illustration 25.**

In a group of 50 people, 30 speak Hindi, 22 speak English and 8 speak neither Hindi nor English. Find how many can speak both English & Hindi.

**Solution:**

$$n(H)=30, n(E)=22, n(H \cup E)'=8$$

$$n(H \cup E)=50-8=42$$

$$\text{Now } n(H \cup E)=n(H)+n(E)-n(HE)$$

$$\begin{aligned} \text{Or } n(HE) &= n(H)+n(E)-n(H \cup E) \\ &= 30+22-42=10 \end{aligned}$$

**Illustration 26.**

Out of 20 members in a family, 11 like tea and 14 like coffee. Assume that each one likes at least one of the two drinks. Find how many like both coffee and tea.

**Solution:**

$$n(T)=11, n(C)=14, n(T \cup C)=20$$

$$\begin{aligned} n(T \cap C) &= n(T)+n(C)-n(T \cup C) \\ &= 11+14-20=5 \end{aligned}$$

**Illustration 27.**

In a class of 80 students, 35% play only cricket, 45% only Tennis. How many play Cricket.

**Solution:**

$$\begin{aligned} n(C \cup T) &= n(C-T)+n(T-C)+n(TC) \\ 100 &= 35+45+n(TC) \end{aligned}$$

$$\begin{aligned} \text{Or } 100-80 &= n(TC) = 20\% \\ n(C) &= n(C-T)+n(TC) \\ &= 35+20 = 55\% \end{aligned}$$

$$\text{No. of students play Cricket} = 80 \times 55\% = 44$$

**Illustration 28.**

If  $A = (0, 2, 4, 5, 7)$ ,  $B = (0, 4, 6, 11, 17)$ , find  $n(A-B)$

**Solution:**

$$A-B = (2, 5, 7)$$

$$n(A-B) = 3$$

**Illustration 29.**

If  $A = (1, 2, 3, 4)$ ,  $B = (3, 5, 7)$ , find  $A \Delta B$ .

**Solution:**

$$\begin{aligned} A \Delta B &= (A-B) \cup (B-A) \\ &= (1, 2, 4) \cup (5, 7) \\ &= (1, 2, 4, 5, 7) \end{aligned}$$

# Indices and Logarithms (Basic Concepts)

## 2.2

### INDICES

The power or exponent of any number or a variable is known as Index. Plural of Index is Indices, Indices provide a compact algebraic notation for repeated multiplication. For solving expression with Indices, the law of exponents is applied. In the expression  $x^5$ , the power 5 is known as Index and  $x$  is known as Base.

For Example,

(a)  $2^3 = 2 \times 2 \times 2 = 8$

(b)  $(2^2)^2 = (2 \times 2)^2 = 4^2 = 4 \times 4 = 16$

1. Find the value of 1.25 to the power of 5.

**Solution:**

1.25 to the power of 5 is written as

$$(1.25)^5$$

$$= (1.25) \times (1.25) \times (1.25) \times (1.25) \times (1.25)$$

$$= 3.05175$$

Thus, 1.25 to the power of 5 = 3.05. (Approx).

2. Find the value of  $(2.5)^5$

**Solution:**

$$(2.5)^5 = (2.5) \times (2.5) \times (2.5) \times (2.5) \times (2.5)$$

$$= 244.14062$$

Thus,  $(2.5)^6 = 244.14$ . (Approx).

**Root of a number:**

If  $x^n = a$  then,  $x$  is called  $n^{\text{th}}$  root of  $a$

$$x = \sqrt[n]{a} \text{ Or, } x = (a)^{1/n}$$

For example:  $2^4 = 16$  or  $2 = \sqrt[4]{16}$

**Basic rules of Indices:**

$$1. a^m \times a^n = a^{m+n}$$

$$\text{For example: } 5^3 \times 5^2 = 5^{3+2} = 5^5$$

$$2. \frac{a^m}{a^n} = a^{m-n}$$

$$\text{For example: } \frac{5^3}{5^2} = 5^{3-2} = 5$$

$$3. \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$\text{For Example: } \left(\frac{6}{5}\right)^2 = \frac{6^2}{5^2}$$

$$4. (ab)^m = a^m \times b^m$$

$$\text{For example: } (5 \times 4)^2 = 5^2 \times 4^2 = 25 \times 16$$

$$5. \frac{1}{a^m} = a^{-m}$$

$$\text{For example: } \frac{1}{5^2} = 5^{-2}$$

$$6. \sqrt[n]{a} = a^{1/n}$$

$$\text{For example: } \sqrt[5]{6} = 6^{1/5}$$

$$7. a^0 = 1$$

$$\text{For example: } 6^0 = 1$$

8. If in an equation bases on both sides are same, powers can be equated.

$$\text{For example: If } 2^2 = 2^x \text{ then } x = 2$$

9. If in an equation powers or indices on both sides are same, bases can be equated.

$$\text{For example: If } x^2 = 2^2 \text{ then } x = 2$$

### Illustration 30.

Evaluate:  $13^3$ .

**Solution:**

$$13^3$$

$$= 13 \times 13 \times 13 = 2197.$$

**Illustration 31.**

If  $a^x = \left(\frac{a}{k}\right)^y = k^m$  and  $a \neq 1$ , prove that  $\frac{1}{x} - \frac{1}{y} = \frac{1}{m}$

**Solution:**

$$a^x = \frac{a^y}{k^y}$$

$$\text{Or, } a^x k^y = a^y$$

$$\text{Or, } k^y = a^{y-x}$$

$$\text{Or, } k = (a^{y-x})^{\frac{1}{y}}$$

$$\text{Or, } k = a^{\frac{y-x}{y}}$$

$$\text{Or, } k^m = \left(a^{\frac{y-x}{y}}\right)^m$$

$$\text{Or, } a^x = a^{\frac{(y-x)m}{y}}$$

$$\text{Or, } x = \frac{(y-x)m}{y}$$

$$\text{Or, } \frac{x}{m} = 1 - \frac{x}{y}$$

$$\text{Or, } \frac{1}{x} - \frac{1}{y} = \frac{1}{m} \text{ (Proved)}$$

**Illustration 32.**

If  $\frac{x}{y} = \frac{z}{w}$  show that  $\frac{x^m + y^m + z^m + w^m}{x^{-m} + y^{-m} + z^{-m} + w^{-m}} = (xyzw)^{\frac{m}{2}}$

**Solution:**

Let  $\frac{x}{y} = \frac{z}{w} = k$ , then  $x = yk, z = wk$

$$\frac{x^m + y^m + z^m + w^m}{x^{-m} + y^{-m} + z^{-m} + w^{-m}} = \frac{y^m k^m + y^m + w^m k^m + w^m}{y^{-m} k^{-m} + y^{-m} + w^{-m} k^{-m} + w^{-m}}$$

$$= \frac{(y^m + w^m)(k^m + 1)}{(y^{-m} + w^{-m})(k^{-m} + 1)}$$

$$= \frac{(y^m + w^m)(k^m + 1)}{\left(\frac{1}{y^m} + \frac{1}{w^m}\right)\left(\frac{1}{k^m} + 1\right)}$$

$$= k^m y^m w^m$$

$$\text{Also } (xyzw)^{\frac{m}{2}} = (yk \cdot y \cdot wk \cdot w)^{m/2} = (y^2 k^2 w^2)^{m/2} = y^m k^m w^m$$

So  $\frac{x^m + y^m + z^m + W^m}{x^{-m} + y^{-m} + z^{-m} + w^{-m}} = (xyzw)^{\frac{m}{2}}$  (Proved)

**Illustration 33.**

Solve:  $6^{2x+4} = 3^{3x} \cdot 2^{x+8}$ .

**Solution:**

We have  $6^{2x+4} = 3^{3x} \cdot 2^{x+8}$

Or,  $2^{2x+4} \cdot 3^{2x+4} = 3^{3x} \cdot 2^{x+8}$

Or,  $\frac{2^{2x+4}}{2^{x+8}} = \frac{3^{3x}}{3^{2x+4}}$

Or,  $2^{2x+4-x-8} = 3^{3x-2x-4}$

Or,  $2^{x-4} = 3^{x-4}$

Or,  $\left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^4$  Or,  $x = 4$

**Illustration 34.**

Simplify  $\frac{x^{m+2n} \cdot x^{3m-8n}}{x^{5m-6n}}$

**Solution:**

$\frac{x^{m+2n} \cdot x^{3m-8n}}{x^{5m-6n}} = x^{m+2n+3m-8n-5m+6n}$

$= x^{4m-6n-5m+6n}$

$= x^{-m} = \frac{1}{x^m}$

**Illustration 35.**

Solve:  $6^x \left(\frac{2}{3}\right)^y - 3 \cdot 2^{x+y} - 8 \cdot 3^{x-y} + 24 = 0, xy = 2$

**Solution:**

$6^x \left(\frac{2}{3}\right)^y - 3 \cdot 2^{x+y} - 8 \cdot 3^{x-y} + 24 = 0$

Or,  $(2 \cdot 3)^x \left(\frac{2}{3}\right)^y - 3 \cdot 2^{x+y} - 8 \cdot 3^{x-y} + 24 = 0$

Or,  $2^x \cdot 3^x \left(\frac{2}{3}\right)^y - 3 \cdot 2^{x+y} - 8 \cdot 3^{x-y} + 24 = 0$

Or,  $2^x \cdot 2^y \cdot 3^x \cdot \frac{1}{3^y} - 3 \cdot 2^{x+y} - 8 \cdot 3^{x-y} + 24 = 0$

Or,  $2^{x+y} \cdot 3^{x-y} - 3 \cdot 2^{x+y} - 8 \cdot 3^{x-y} + 24 = 0$

$$\text{Or, } 2^{x+y} \cdot 3^{x-y} - 3 \cdot 2^{x+y} = 8 \cdot 3^{x-y} - 24$$

$$\text{Or, } 2^{x+y} (3^{x-y} - 3) = 8(3^{x-y} - 3)$$

$$\text{Or, } 2^{x+y} = 8 \quad [\text{Assuming } 3^{x-y} - 3 \neq 0]$$

$$\text{Or, } 2^{x+y} = 2^3$$

$$\text{Or, } x + y = 3$$

$$\text{Or, } x = 3 - y$$

Putting this value of  $x$  in  $xy = 2$  we get

$$(3 - y)y = 2$$

$$\text{Or, } 3y - y^2 = 2$$

$$\text{Or, } y^2 - 3y + 2 = 0$$

$$\text{Or, } y = 2 \text{ or } 1$$

$$\text{So } x = 1 \text{ or } 2$$

### Illustration 36.

Obtain the simplest value of  $[1 - \{1 - (1 - x^3)^{-1}\}^{-1}]^{-\frac{1}{3}}$  when  $x = 1$

**Solution:**

$$[1 - \{1 - (1 - x^3)^{-1}\}^{-1}]^{-\frac{1}{3}}$$

$$= \left[ 1 - \frac{1}{1 - (1 - x^3)^{-1}} \right]^{-\frac{1}{3}}$$

$$= \left[ \frac{1 - (1 - x^3)^{-1} - 1}{1 - (1 - x^3)^{-1}} \right]^{-\frac{1}{3}}$$

$$= \left( \frac{-\frac{1}{1 - x^3}}{1 - \frac{1}{1 - x^3}} \right)^{-\frac{1}{3}}$$

$$= \left( \frac{1}{x^3} \right)^{-\frac{1}{3}}$$

$$= (x^3)^{\frac{1}{3}} = x = 1 \quad [\text{As } x = 1 \text{ (given)}]$$

### Illustration 37.

If  $x = 3^{1/3} + 3^{-1/3}$  prove that  $3x^3 - 9x = 10$

**Solution:**

$$x = 3^{1/3} + 3^{-1/3}$$

Cubing both sides we get

$$x^3 = (3^{1/3})^3 + (3^{-1/3})^3 + 3 \cdot 3^{1/3} \cdot 3^{-1/3} (3^{1/3} + 3^{-1/3})$$

$$\text{Or, } x^3 = 3 + 3^{-1} + 3 \cdot 3^{\frac{1}{3}-\frac{1}{3}}(x) \quad [\text{As } x = 3^{\frac{1}{3}} + 3^{-\frac{1}{3}} \text{ (Given)}]$$

$$\text{Or, } x^3 = 3 + \frac{1}{3} + 3 \cdot 1 \cdot x$$

$$\text{Or, } 3 \cdot x^3 = 9 + 1 + 9x$$

$$\text{Or, } 3x^3 - 9x = 10 \text{ (Proved)}$$

**Illustration 38.**

If  $(2.381)^x = (0.2381)^y = 10^z$  prove that  $\frac{1}{x} = \frac{1}{y} + \frac{1}{z}$

**Solution:**

$$\text{Let } (2.381)^x = (0.2381)^y = 10^z = K$$

$$\text{So } 2.381 = K^{\frac{1}{x}} \text{ \& } 0.2381 = K^{\frac{1}{y}} \text{ \& } 10 = K^{\frac{1}{z}}$$

$$\text{Or, } \frac{2381}{1000} = K^{\frac{1}{x}}$$

$$\text{Or, } 2381 = 10^3 K^{\frac{1}{x}} \dots \dots \dots (1)$$

$$0.2381 = K^{\frac{1}{y}}$$

$$\text{Or, } \frac{2381}{10000} = K^{\frac{1}{y}}$$

$$\text{Or, } 2381 = 10^4 K^{\frac{1}{y}} \dots \dots \dots (2)$$

(2) ÷ (1) gives

$$\frac{2381}{2381} = \frac{10^4 K^{\frac{1}{y}}}{10^3 K^{\frac{1}{x}}}$$

$$\text{Or, } 1 = \frac{10 \cdot K^{\frac{1}{y}}}{K^{\frac{1}{x}}}$$

$$\text{Or, } K^{\frac{1}{x}} = 10 K^{\frac{1}{y}}$$

$$\text{Or } K^{\frac{1}{x}} = k^{\frac{1}{z}} \cdot K^{\frac{1}{y}}$$

$$\text{So } \frac{1}{x} = \frac{1}{y} + \frac{1}{z} \text{ (Proved)}$$

**Illustration 39.**

If  $x = 3^{1/4} + 3^{-1/4}$  &  $y = 3^{1/4} - 3^{-1/4}$  prove that  $3(x^2 + y^2)^2 = 64$

**Solution:**

$$x = 3^{1/4} + 3^{-1/4}$$

Squaring both sides we have

$$x^2 = (3^{1/4} + 3^{-1/4})^2 = 3^{1/2} + 3^{-1/2} + 2.3^{1/4}.3^{-1/4}$$

$$\text{Or, } x^2 = 3^{1/2} + 3^{-1/2} + 2 \dots \dots \dots (1)$$

Also

$$y = 3^{1/4} - 3^{-1/4}$$

Squaring both sides we have

$$y^2 = (3^{1/4} - 3^{-1/4})^2 = 3^{1/2} + 3^{-1/2} - 2.3^{1/4}.3^{-1/4}$$

$$\text{Or, } y^2 = 3^{1/2} + 3^{-1/2} - 2 \dots \dots \dots (2)$$

(1) + (2) gives

$$x^2 + y^2 = 3^{1/2} + 3^{-1/2} + 3^{1/2} + 3^{-1/2}$$

$$\text{Or, } x^2 + y^2 = 2(3^{1/2} + 3^{-1/2})$$

Squaring both sides we get

$$(x^2 + y^2)^2 = 4(3 + 3^{-1} + 2.3.3^{-1})$$

$$\text{Or, } (x^2 + y^2)^2 = 4\left(3 + \frac{1}{3} + 2\right) = 4\left(\frac{9+1+6}{3}\right)$$

$$\text{Or, } 3(x^2 + y^2)^2 = 64 \text{ (Proved)}$$

**Illustration 40.**

If  $a + b + c = 0$ , simplify  $x^{a^2b^{-1}c^{-1}} x^{a^{-1}b^2c^{-1}} x^{a^{-1}b^{-1}c^2}$

**Solution:**

$$x^{a^2b^{-1}c^{-1}} x^{a^{-1}b^2c^{-1}} x^{a^{-1}b^{-1}c^2}$$

$$= x^{a^2/bc} \cdot x^{b^2/ac} \cdot x^{c^2/ab}$$

$$= x^{a^2/bc + b^2/ac + c^2/ab}$$

$$= x^{a^3 + b^3 + c^3 / abc}$$

$$= x^{[a^3+b^3+c^3-3abc+3abc]/abc}$$

Now  $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^3 + b^3 + c^3 - ab - bc - ca) = 0$  [As  $a + b + c = 0$  (given)]

So  $x^{[a^3+b^3+c^3-3abc+3abc]/abc} = x^{3abc/abc} = x^3$

**Illustration 41.**

Express as positive indices –

(i)  $x^{-\frac{1}{3}}$

(ii)  $2x^{-\frac{2}{5}} \cdot x^{-\frac{1}{3}}$

(iii)  $x^{\frac{2}{3}} \div a^{-\frac{1}{5}}$

**Solution:**

(i)  $x^{-\frac{1}{3}} = \frac{1}{x^{\frac{1}{3}}}$ ;

(ii)  $2x^{-\frac{2}{5}} \cdot x^{-\frac{1}{3}} = \frac{2}{x^{\frac{2}{5}}} \times \frac{1}{x^{\frac{1}{3}}}$ ;

(iii)  $x^{\frac{2}{3}} \div a^{-\frac{1}{5}} = \frac{x^{\frac{2}{3}}}{a^{-\frac{1}{5}}} = x^{\frac{2}{3}} \times a^{\frac{1}{5}}$

**Illustration 42.**

Find the value of (i)  $\left(\frac{1}{81}\right)^{\frac{3}{4}}$ ; (ii)  $(243)^{-\frac{1}{5}}$

**Solution:**

(i)  $\left(\frac{1}{81}\right)^{-\frac{3}{4}} = (81)^{\frac{3}{4}} = (3^4)^{\frac{3}{4}} = 3^3 = 27$

(ii)  $(243)^{-\frac{1}{5}} = (3^5)^{-\frac{1}{5}} = 3^{-1} = \frac{1}{3}$

**Illustration 43.**

Show that  $\frac{2(3^{n+1}) + 7(3^{n-1})}{3^{n+2} - 2\left(\frac{1}{3}\right)^{1-n}} = 1$

**Solution:**

$$2(3^{n+1}) + 7(3^{n-1}) = 2 \cdot (3^n \cdot 3) + 7 \cdot (3^n \cdot 3^{-1}) = 3^n \left(2 \cdot 3 + 7 \cdot \frac{1}{3}\right) = 3^n \cdot \frac{25}{3}$$

$$3^{n+2} - 2\left(\frac{1}{3}\right)^{1-n} = 3^n \cdot 3^2 - 2 \cdot (3^{-1})^{1-n} = 3^n \cdot 3^2 - 2 \cdot 3^n \cdot 3^{-1} = 3^n \left(9 - \frac{2}{3}\right) = 3^n \frac{25}{3}$$

$$\text{So } \frac{2(3^{n+1}) + 7(3^{n-1})}{3^{n+2} - 2\left(\frac{1}{3}\right)^{1-n}} = \frac{3^n \cdot \frac{25}{3}}{3^n \frac{25}{3}} = 1 \text{ (Proved)}$$

**Illustration 44.**

$$\text{Simplify } \frac{\left(p + \frac{1}{q}\right)^p \cdot \left(p - \frac{1}{q}\right)^q}{\left(q + \frac{1}{p}\right)^p \cdot \left(q - \frac{1}{p}\right)^q}$$

**Solution:**

$$\begin{aligned} & \frac{\left(p + \frac{1}{q}\right)^p \cdot \left(p - \frac{1}{q}\right)^q}{\left(q + \frac{1}{p}\right)^p \cdot \left(q - \frac{1}{p}\right)^q} = \frac{\left(\frac{pq+1}{q}\right)^p \cdot \left(\frac{pq-1}{q}\right)^q}{\left(\frac{pq+1}{p}\right)^p \cdot \left(\frac{pq-1}{p}\right)^q} = \frac{\left(\frac{pq+1}{q}\right)^p \cdot \left(\frac{pq-1}{q}\right)^q}{\left(\frac{pq+1}{p}\right)^p \cdot \left(\frac{pq-1}{p}\right)^q} \\ & = \frac{(pq+1)^p}{q^p} \cdot \frac{p^p}{(pq+1)^p} \cdot \frac{(pq-1)^q}{q^q} \cdot \frac{p^q}{(pq-1)^q} \\ & = \left(\frac{p}{q}\right)^{p+q} \end{aligned}$$

**Illustration 45.**

$$\text{Show that } \left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2} \times \left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2} \times \left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2} = 1$$

**Solution:**

$$\begin{aligned} & \left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2} \times \left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2} \times \left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2} \\ & = x^{(a-b)(a^2+ab+b^2)} \times x^{(b-c)(b^2+bc+c^2)} \times x^{(c-a)(c^2+ca+a^2)} \\ & = (x)^{a^3-b^3} \cdot (x)^{b^3-c^3} \cdot (x)^{c^3-a^3} = x^{a^3-b^3+b^3-c^3+c^3-a^3} \\ & = x^0 = 1 \end{aligned}$$

**Illustration 46.**

Find the value of  $a + b + c$ , if  $x^{\frac{1}{a}} = y^{\frac{1}{b}} = z^{\frac{1}{c}}$  is satisfying  $xyz = 1$

**Solution:**

$$\text{Let } x^{\frac{1}{a}} = y^{\frac{1}{b}} = z^{\frac{1}{c}} = k$$

$$\text{So } x = k^a, y = k^b, z = k^c$$

$$\text{So } xyz = k^a \cdot k^b \cdot k^c = 1$$

$$\text{Or, } k^{a+b+c} = k^0 \text{ Or } a + b + c = 0$$

**Illustration 47.**

$$\text{Solve } 2^{x+3} + 2^{x+1} = 320$$

**Solution:**

$$2^{x+3} + 2^{x+1} = 320$$

$$\text{Or, } 2^{x+1}(2^2 + 1) = 320$$

$$\text{Or, } 2^{x+1} = \frac{320}{5} = 64 = 2^6$$

$$\text{Or, } x + 1 = 6. \text{ So } x = 5$$

**Illustration 48.**

If  $a^x = b$ ,  $b^y = c$ ,  $c^z = a$ . find 'xyz'.

**Solution:**

$$c^z = a \quad \text{or, } b^{yz} = a$$

$$\text{or, } a^{xyz} = a^1$$

$$\text{or, } xyz = 1$$

**Illustration 49.**

If  $a^p = b^q = (ab)^{pq}$ , find 'p+q'.

**Solution:**

$$a^p = b^q = (ab)^{pq} = K.$$

$$\text{Or, } a = K^{\frac{1}{p}}, b = K^{\frac{1}{q}}, ab = K^{\frac{1}{pq}}$$

$$\text{Now } a \cdot b = ab$$

$$\text{Or, } K^{\frac{1}{p}} \times K^{\frac{1}{q}} = K^{\frac{1}{pq}}$$

$$\text{Or, } K^{\frac{1}{p} + \frac{1}{q}} = K^{\frac{1}{pq}}$$

$$\text{Or, } \frac{1}{p} = \frac{1}{q} = \frac{1}{pq} \quad \text{Or, } \frac{q+p}{pq} = \frac{1}{pq}$$

$$\text{Or, } p + q = 1$$

## LOGARITHM

### Definition

If  $a^x = N$  and  $a > 0, N > 0$  then  $x = \log_a N$  and read as logarithm of  $N$  to the base ' $a$ ' or in other words  $\log N$  to the base ' $a$ '.

So logarithm of a positive number  $N$  signifies that for a given base ' $a$ ' it is the power to which ' $a$ ' must be raised to produce the positive number ' $N$ '.

For example,

Since  $5^3 = 125$  we express the corresponding logarithm as  $\log_5 125 = 3$ . This signifies when 5 is raised to the power of 3 the value would be 125.

### Understanding Common and Natural Logarithms

Common Logarithms are the logarithms for which the base is considered as 10. For any calculation with logarithms, if base is taken as 10, the logarithms are considered to be Common Logarithms. In such scenarios, the base to a logarithm is not generally mentioned but is assumed to be 10.

When the base of Logarithm is  $e$  (where  $e$  means exponential and having value 2.718) then it is called Natural Logarithm. Its extensive use is seen in Calculus.

For example,

If  $\log 25$  is given, we consider it to be  $\log_{10} 25$ .

Also  $\log_e 25$  is written as  $\ln 25$ .

### Laws followed for Logarithms

1. Logarithm of the product of the numbers would be equal to the sum of their logarithms.

$$\log_a (mn) = \log_a m + \log_a n$$

For Example,

$$\log_5 (14) = \log_5 (2 \times 7) = \log_5 2 + \log_5 7$$

2. Logarithms of the quotient of two numbers would be equal to the difference of their logarithms.

$$\log_a \left( \frac{m}{n} \right) = \log_a m - \log_a n$$

For Example,

$$\log_5 (3.2) = \log_5 \frac{32}{10} = \log_5 32 - \log_5 10$$

3. Logarithm of any number raised to a power would be equal to the product of the logarithm of the number and the power.

$$\log_a (m)^n = n \times \log_a m$$

For example,

$$\log_2 (3)^4 = 4 \times \log_2 3$$

4. For any logarithm where base is same as the number, the value of logarithm would always be one (1). The

value of base or number should not be equal to 1 and should be a positive number ( $>0$ ).

$$\log_p p = 1$$

For example,

$$\begin{aligned}\log_2 2 &= 1 \\ \log_{10} 10 &= \log_{10} 10 = 1\end{aligned}$$

5. For any logarithm where number is 1 and the base is a positive number, then value would always be zero.

$$\log_p 1 = 0$$

For example,

$$\log_4 1 = 0$$

6. The base of a logarithm for a number would be equal to the reciprocal of the logarithm with the number replaced by base.

$$\log_r s = \frac{1}{\log_s r}$$

For example,

$$\log_5 8 = \frac{1}{\log_8 5}$$

7. If logarithm with base of a number is raised to its' power, the logarithm of the number raised to the power would be equal to the number.

$$s^{\log_s r} = r$$

For example,

$$8^{\log_8 16} = 16$$

### Important points to be kept in mind

#### Change of Base for Logarithm

The logarithm of a number with a base can be changed with the introduction of the base and number splitted into two expressions of logarithms. The base and the number would be same with a new variable introduced both as base and number to the new logarithmic expressions.

$$\log_r s = \log_r s \times \log_r t$$

#### Illustration 50.

Express in simplified form:  $\log_2 49$

**Solution:**

$$\begin{aligned}\log_2 49 &= \log_2 (7 \times 7) \\ &= \log_2 7 + \log_2 7 \\ &= 2 \times \log_2 7\end{aligned}$$

#### Illustration 51.

Express in simplified form:  $\log_2 \frac{97}{7}$

**Solution:**

$$\log_2 \frac{97}{7} = \log_2 97 - \log_2 7$$

**Illustration 52.**Express as sum of two logarithms:  $\log_2(15 \times 3)$ **Solution:**

$$\log_2(15 \times 3) = \log_2 15 + \log_2 3$$

**Illustration 53.**Express as difference of two logarithms:  $\log\left(\frac{50}{2}\right)$  with base 2.**Solution:**

$$\log\left(\frac{50}{2}\right) \text{ with base 2 would be written as : } \log_2\left(\frac{50}{2}\right)$$

$$\log_2\left(\frac{50}{2}\right) = \log_2 50 - \log_2 2$$

$$= \log_2 50 - 1$$

**Illustration 54.**Find the value of:  $\log_2 64$ **Solution:**

$$\log_2 64 = \log_2 2^6$$

$$= 6 \times \log_2 2$$

$$= 6 \times 1 = 6$$

**Illustration 55.**Find the simplified form:  $\log_3 4$  Write logarithms by following the way it is written here**Solution:**

$$\log_3^4 = \log_3^{2^2} = 2 \log_3^2$$

**Illustration 56.**Express as quotient of two logarithms with base 10:  $\log_8 83$ **Solution:**

$$\log_8 83 = \frac{\log_{10} 83}{\log_{10} 8}$$

**Illustration 57.**Express with base value as 3:  $\log_2 3$

**Solution:**

$$\log_2 3 = \frac{1}{\log_3 2}$$

**Illustration 58.**Express with base value as 3 after simplification :  $\log_{12} 3$ **Solution:**

$$\begin{aligned}\log_{12} 3 &= \frac{1}{\log_3 12} \\ &= \frac{1}{\log_3 12} = \frac{1}{\log_3 (3 \times 4)} \\ &= \frac{1}{\log_3 (3 \times 4)} = \frac{1}{\log_3 3 + \log_3 4} \\ &= \frac{1}{1 + \log_3 4}\end{aligned}$$

**Illustration 59.**Find the value of:  $3^{\log_3 9}$ **Solution:**

$$3^{\log_3 9} = 9$$

**Illustration 60.**Find the logarithm of 2025 to the base  $3\sqrt{5}$ **Solution:**Let  $x$  be the required number; then  $(3\sqrt{5})^x = 2025 = 3^4 5^2 = (3\sqrt{5})^4 \therefore x = 4$  is the required number**Illustration 61.**The logarithm of a number of the base  $\sqrt{2}$  is  $k$ . What is its logarithm to the base  $2\sqrt{2}$  ?**Solution:**

$$\text{Let } (\sqrt{2})^k = N$$

$$2\sqrt{2} = 2 \times 2^{\frac{1}{2}} = 2^{\frac{3}{2}}$$

$$\text{So } \sqrt{2} = 2^{\frac{1}{2}} = \left(2^{\frac{3}{2}}\right)^{\frac{1}{3}} = (2\sqrt{2})^{\frac{1}{3}}$$

$$\text{So } (\sqrt{2})^k = (2\sqrt{2})^{\frac{k}{3}} = N$$

So the required number is  $\frac{k}{3}$ **Illustration 62.**If  $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$  find  $x$

**Solution:**

$$\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$$

$$\text{Or, } \log_{16} x \log_2 16 + \log_{16} x \log_4 16 + \log_{16} x = \frac{21}{4}$$

$$\text{Or, } \log_{16} x (\log_2 16 + \log_4 16 + 1) = \frac{21}{4}$$

$$\text{Or, } \log_{16} x (4 + 2 + 1) = \frac{21}{4}$$

$$\text{Or, } \log_{16} x = \frac{3}{4}$$

$$\text{Or, } x = (16)^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 8$$

**Illustration 63.**

If  $p = \log_{10} 20$  and  $q = \log_{10} 25$ , find  $x$  and such that  $2 \log_{10}(x+1) = 2p - q$

**Solution:**

$$2p - q = 2 \log_{10} 20 - \log_{10} 25 = \log_{10} 20^2 - \log_{10} 25 = \log_{10} \left( \frac{400}{25} \right) = \log_{10} 16$$

$$\text{Or, } 2 \log_{10}(x+1) = \log_{10}(x+1)^2 = \log_{10} 16$$

$$\text{Or, } (x+1)^2 = 16$$

$$\text{Or, } x+1 = \pm 4$$

$$\text{Or, } x = 3, -5$$

If  $x = -5$  then  $x+1 = -4$ . In that case  $2 \log_{10}(x+1) = 2 \log_{10}(-4)$  which is impossible. So  $x \neq -5$

Required value of  $x$  is 3

**Illustration 64.**

If  $x = \log_{2a} a$ ,  $y = \log_{3a} 2a$ ,  $z = \log_{4a} 3a$  show that  $xyz + 1 = 2yz$

**Solution:**

$$LHS = xyz + 1 = \log_{2a} a \times \log_{3a} 2a \times \log_{4a} 3a + 1$$

$$= (\log_{10} a \times \log_{2a} 10) \times (\log_{10} 2a \times \log_{3a} 10) \times (\log_{10} 3a \times \log_{4a} 10) + 1$$

$$= \frac{\log_{10} a}{\log_{10} 2a} \times \frac{\log_{10} 2a}{\log_{10} 3a} \times \frac{\log_{10} 3a}{\log_{10} 4a} + 1$$

$$= \frac{\log_{10} a}{\log_{10} 4a} + 1$$



$$\text{Or, } ab = (ab)^{\left(\frac{1}{x} + \frac{1}{y}\right)}$$

$$\frac{1}{x} + \frac{1}{y} = 1$$

$$\text{Or, } \frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = 1 \text{ (Proved)}$$

**Illustration 67.**

Find logarithm of 0.0001 to the base 0.01.

**Solution:**

$$\text{Log}_{(0.01)}^{(0.0001)} = x$$

$$\text{Or } (0.01)^x = 0.0001$$

$$\text{Or } \left(\frac{1}{100}\right)^x = \frac{1}{10000} = \frac{1}{10^4} = 10^{-4}$$

$$\text{Or } 10^{-2x} = 10^{-4} \quad \text{Or } -2x = -4$$

$$\text{Or } x = 2.$$

**Illustration 68.**

Find logarithm of 0.333 to the base 81.

**Solution:**

$$\log_{81}(0.333) = x \quad \text{Or } (81)^x = .333$$

$$\text{Or, } \left(\frac{1}{3^4}\right)^x = \frac{1}{3} \quad \text{Or } 3^{-4x} = 3^{-1}$$

$$\text{Or, } -4x = -1 \quad \text{Or } x = \frac{1}{4}$$

**Illustration 69.**

$\text{Log}(a^9) + \log a = 10$ . Find 'a'

**Solution:**

$$\log(a^9 \cdot a) = 10 \quad \text{Or } 10^{10} = a^{10} \quad \text{Or } a = 10$$

**Illustration 70.**

Find value of  $\log_2 \log_2 \log_2 16$

**Solution:**

$$\log_2 \log_2 \log_2 16 = \log_2 \log_2 \log_2 2^4 = \log_2 \log_2 4 \log_2 2 = \log_2^2 \log_2^2 = 4 \quad (\log_2^2 = 1)$$

**Illustration 71.**

Find value of  $\log_a^b + \log_a^{b^2} + \dots + \log_a^{b^n}$

**Solution:**

$$\log_a^{b^2} = 2 \log_a^b = \frac{2}{\log_b a^2} = \frac{2}{2 \log_b a} = \log_a^b$$

$$\text{So, } \log_a^b + \log_a^b + \dots + \log_a^b = n \log_a^b$$

**Illustration 72.**

If  $\log t + \log(t-3) = 1$ . Find 't'.

**Solution:**

$$\log t(t-3) = \log 10$$

$$\text{Or, } t^2 - 3t = 10$$

$$\text{Or, } t^2 - 3t - 10 = 0$$

$$\text{Or, } t^2 - 5t + 2t - 10 = 0$$

$$\text{Or, } t(t-5) + 2(t-5) = 0$$

$$\text{Or, } (t-5)(t+2) = 0$$

$$\text{Or, } t-5=0 \quad \text{or} \quad t=5$$

$$\text{Or, } t+2=0 \quad \text{or} \quad t=-2 \text{ (Absurd)}$$

**Illustration 73.**

Find value of  $\log_b^a \log_c^b \times \log_a^c$ .

**Solution:**

$$(\log_b^a \times \log_c^b) \log_a^c = \log_c^a \times \log_a^c = 1.$$

**Illustration 74.**

Find value of  $\log \frac{a^2}{bc} + \log \frac{b^2}{ca} + \log \frac{c^2}{ab}$

**Solution:**

$$\log a^2 - \log bc + \log b^2 - \log ca + \log c^2 - \log ab$$

$$= 2\log a - \log b - \log c + 2\log b - \log c - \log a + 2\log c - \log a - \log b$$

$$= 0.$$

# Permutation and Combinations (Basic Concepts)

## 2.3

### PERMUTATION

#### Fundamental Principle of Counting:

Let travelling from city A to city B is possible either by car or by bus or by train i.e. there are 3 ways of going from city A to city B and with each of these 3 ways there are associated 2 ways (by auto or by rickshaw) of going from city B to city C. Hence possible ways of going from A to C via B can be  $3 \times 2 = 6$

Above concept can be generalised as –

If there are ‘m’ ways of doing a job and with each of these ‘m’ ways there are associated ‘n’ ways of doing another job then the two jobs can be done in  $(m \times n)$  number of ways.

#### Understanding the meaning of ‘n’ factorial

Taking any number as ‘n’. The Factorial of such number would be ‘n’ factorial and is denoted as  $n!$  or  $\lfloor n$

This means product of all integers from n upto 1. So  $n! = n(n-1)(n-2) \dots 3,2,1$ . It can be mentioned that factorials are possible only for positive integers.

For Example,

Factorial of 3: is expressed as  $3!$  and it is calculate as  $3! = 3 \times 2 \times 1$

$$11! = 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

Important results related to factorial are –

$$(i) 0! = 1, (ii) 1! = 1, (iii) 2n! = 2n(2n-1)(2n-2)(2n-3) \dots 3,2,1$$

#### Illustration 75.

Find the value of ‘Factorial of 7’

**Solution:** Factorial of 7 would be written as  $7!$

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040.$$

#### Permutation of n different objects taking r at a time ( $r \leq n$ )

The word permutation means Arrangement. Total number of permutations or arrangements that can be made out of ‘n’ different objects taking ‘r’ at a time (when  $r \leq n$ ) is given as  $n_{Pr}$ . The formula for it is given below:

**Formula:**

$$n_{p_r} = \frac{n!}{(n-r)!}$$

$$n_{p_r} = n \times (n-1) \times (n-2) \times (n-3) \dots \times (n-r+1)$$

Where, 'n' and 'r' are both positive integers.

Permutations can also be represented as

$$P(n,r), Pn,r$$

Permutations are made for arrangements of objects in linear order or rearrangements of such already ordered set. It is a type of arrangement made in definite order.

**Illustration 76.**

With 5,000 hospitals to be built in the state of Goa, only 2,500 hospitals would have ICU Beds. Officials are required to select 2,500 hospitals for installation of ICU Beds. What would be the value of 'n' and 'r' in the above situation?

**Solution:**

In the given question, the total number of hospitals to build are 5,000 and ICU Beds to be installed in only 2,500 hospitals out of those 5,000 hospitals built.

For the purpose of Permutation, 'n' stands for the total number of objects which are available for selection. Hence, in the given question, total hospitals to be built are 5,000 hospitals. So, the value of 'n' would be 5,000.

As 2500 hospitals to be selected, the value of 'r' should be 2500.

**Classification of Permutations:**

- (i) When Repetition is not allowed
  - (ii) When Repetition is allowed
  - (iii) Permutation in cases where distinction of objects is not possible
  - (iv) Circular permutation
- (i) When Repetition is not allowed

When repetition is not allowed, with every object being selected, the availability of the remaining objects do decrease with the selection of objects from the availability. Basically, the total number of objects available for selection decreases after every selection is done.

**Illustration 77.**

In how many ways can 5 candidates be screened for 3 vacancies?

**Solution:**

There are 3 vacancies and 5 candidates.

First vacancy can be filled up by selecting 1 out of 5 in 5 ways.

So there remains 2 vacancies to be filled up by selecting 2 from the remaining.

Hence second vacancy can be filled up in 4 ways.

Similarly third vacancy can be filled up by selecting 1 out of the remaining 3 in 3 ways.

So, the number of ways of filling up 3 vacancies from 5 candidates =  $5 \times 4 \times 3 = 60$

Therefore, there are 60 ways in which 5 candidates can be screened for 3 vacancies.

(ii) When Repetition is allowed

When the objects to be selected can have repetition without any restriction. In simple words, ‘ $n$ ’ different objects could be taken ‘ $r$ ’ at a time.

Formula:  $n^r$

### Illustration 78.

In how many ways 3 cheques be given to 4 employees when each employee is eligible to receive any number of cheques?

#### Solution:

All the 3 cheques can be given to all 4 employees. Even after the 1<sup>st</sup> cheque is given, the rest 2 cheques can also be given to any employee out of 4 employees. So, the total number of ways in which 3 cheques can be given to 4 employees –

$$4 \times 4 \times 4 = 4^3 = 64 \text{ ways.}$$

(iii) Permutation of cases where distinction of objects is not possible

When objects are similar in nature and cannot be distinguished among themselves or the identical distinction of any object from the available objects is not possible due to objects being identical, in such cases if permutation would be done considering all alike objects to be one single object. Thus, while finding number of possible permutations, the denominator would be divided by the number of permutation of alike objects as well.

The permutation of ‘ $n$ ’ objects of which ‘ $p$ ’ are alike of one kind and ‘ $q$ ’ are alike of another kind is  $\frac{n!}{p! \cdot q!}$

### Illustration 79.

Determine the arrangements which could be made from the letters of the word “APPLE”?

#### Solution:

The word “APPLE” has 5 letters with 2 P’s and 1 each of A, L and E. Thus, required the number of arrangements would be

$$\frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{2} = 60$$

(iv) Circular Permutation

When the objects are arranged in a circular formation then that is called Circular Permutation.

- No. of circular permutations of ‘ $n$ ’ different objects taking all at a time is  $(n-1)!$
- No. of circular permutations of ‘ $n$ ’ different taking all at a time is  $(n-1)!/2$  when clockwise or anti clockwise arrangements are not different.

### Illustration 80.

In how many ways 8 boys can form a ring?

**Solution:**

Keeping one boy fixed in any position, remaining 7 boys can be arranged in 7! Ways.

Hence, the required no. of ways = 7! = 7.6.5.4.3.2.1 = 5040.

**Illustration 81.**

In how many ways 8 different beads can be placed in a necklace?

**Solution:**

8 beads can be arranged in 7! Ways. In this 7! Ways, arrangements counting from clockwise and anticlockwise are taken different, but necklace obtained by clockwise permutation will be same as that obtained from anticlockwise. So total arrangement will be half of 7!

Hence, required no. of ways =  $\frac{1}{2} \times 7! = \frac{1}{2} \times 5040 = 2520$

**Illustration 82.**  $\frac{1}{4!} + \frac{1}{5!} = \frac{x}{6!}$  find 'x'

**Solution:**  $\frac{1}{4!} + \frac{1}{5 \times 4!} = \frac{x}{6 \times 5 \times 4!}$

Or,  $1 + \frac{1}{5} = \frac{x}{30}$  (cancelling  $\frac{1}{4!}$ , from both sides)

Or,  $\frac{6}{5} = \frac{x}{30}$  or  $x = 36$

**Illustration 83.** There are 21 boys and 19 girls in a class. In how many ways can one boy and one girl be selected.

**Solution:** There are 21 ways to select a boy. For each of these ways, there are 19 ways to select a girl. Hence required no. of ways =  $21 \times 19 = 399$ .

**Illustration 84.** In how many ways the 1st & 2nd prizes of math & stat be given in a class of 40 students.

**Solution:** 1st & 2nd prizes in math & stat can be given to 40 students in  ${}^{40}P_2$  ways. For each these ways, the 1st & 2nd prizes can be given again in  ${}^{40}P_2$  ways.

Required no. of ways =  ${}^{40}P_2 \times {}^{40}P_2$

**Illustration 85.** How many words can be formed by using the letters of the word 'ALLAHABAD', so that vowels will occupy even places.

**Solution:** 4 vowels (AAAA) can be posted in 4 even places in 1 way.

Then 5 consonants can be arranged among themselves in  $\frac{5!}{2!} = 60$  ways

Required no. of ways =  $60 \times 1 = 60$

**Illustration 86.** How many 3 letter words can be formed by using the letters of the word 'SQUARE'. Where no letter is repeated.

**Solution:**

6	5	4
---	---	---

=  $6 \times 5 \times 4 = 120 = {}^6P_3$

**Illustration 87.** There are 5 doors and 4 windows in a room. How many ways a cat will enter and leave that room.

**Solution:** The cat will enter and leave that room is  $(9 \times 9) = 81$  ways.

## COMBINATION

Combination means selection or choice or formation of a group.

Each of different groups of selection that can be made by taking some or all of the no. of things at a time is called combination.

### Combination of 'n' different objects taking 'r' at a time (when $r \leq n$ )

It is denoted by  $n_{C_r}$  and is defined as  $n_{C_r} = \frac{n!}{(n-r)!r!}$

Here  $n$  = given total no. of things

$C$  = combination or selection

$r$  = no. of things taken at a time

### Illustration 88.

In how many ways a committee of 2 boys can be formed from 4 boys.

**Solution:**

$${}^4C_2 = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{4.3.2.1}{2.1.2.1} = 6$$

### Relation between $n_{Pr}$ and $n_{Cr}$ :

$$n_{Pr} = \frac{n!}{(n-r)!}$$

$$\text{Or, } \frac{n_{Pr}}{r!} = \frac{n!}{(n-r)!r!} = n_{Cr}$$

$$\text{Or, } n_{Pr} = r!n_{Cr}$$

### Properties:

1.  $n_{C_r} = 1$
2.  $n_{C_0} = 1$
3.  $n_{C_r} = n_{C_{n-r}}$

### Proof:

$$\text{R.H.S: } n_{C_{n-r}} = \frac{n!}{(n-n+r)!(n-r)!} = \frac{n!}{(n-r)!r!} = n_{C_r} = \text{L.H.S}$$

4. If  $n_{C_r} = n_{C_s}$  then  $r = s$  and also  $r = n - s$

5. Pascal's law:

If  $n$  and  $r$  be two non-negative integers such that  $1 \leq r \leq n$ . then

$$n_{C_{r-1}} + n_{C_r} = n + 1_{C_r}$$

**Formulae:**

- A) No. of combinations of  $n$  dissimilar things taken  $r$  at a time when ' $p$ ' particular things always occur is  $n - P_{Cr-p}$ ;
- B) No. of combinations of  $n$  dissimilar things taken ' $r$ ' at a time when ' $p$ ' particular things never comes is  $n - p_{Cr}$ ;
- C) Total No. of combinations of  $n$  different things taken 1, 2, 3..... $n$  things at a time is  $2^n - 1$  i.e.  $n_{C_1} + n_{C_2} + n_{C_3} + \dots + n_{C_n} = 2^n - 1$ .

Total no. of combinations of  $n$  different things taken one or more things at a time is  $2^n - 1$ .

**Grouping:**

When it is required to form two groups out of  $(m + n)$  things, ( $m \neq n$ ) so that one group consists of  $m$  things then formation of one group represents the formation of the other group automatically. Hence the number of ways  $m$  thing can be selected from  $(m + n)$  things.

$$m + n_{C_m} = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m+n)!}{m!n!}$$

**Note:**

1. If  $m = n$ , the groups are equal and in this case the number of different ways of subdivision =  $\frac{(2m)!}{(m!)^2}$  since two groups can be interchanged without getting a new subdivision.
2. When  $2m$  things are divided equally between 2 persons, then the number of ways =  $\frac{(2m)!}{(m!)^2}$
3. Now  $(m + n + p)$  things ( $m \neq n \neq p$ ), to be divided in to three groups containing  $m, n, p$  things respectively. ' $m$ ' things can be selected out of  $(m + n + p)$  things in  $(m + n + p_{C_m})$  ways. Thereafter  $n$  things out of the remaining  $(n + p)$  things can be selected in  $(n+p)_{C_n}$  ways. Hence total number of ways of selection to form three groups containing  $m, n$  and  $p$  things =  $(m + n + p)_{C_m} \times (n + p)_{C_n} = \frac{(m+n+p)!}{m!(n+p)!} \times \frac{(n+p)!}{n!p!} = \frac{(m+n+p)!}{m!n!p!}$

**Note:**

1. If now  $m = n = p$ , the groups are equal and in this case, the different ways of subdivision =  $\frac{3m!}{m!m!m!} \times \frac{1}{3!}$  since the three groups of subdivision can be arranged in 3! Ways.

**Note 2:**

If  $3m$  things are divided equally amongst three persons, the number of ways =  $\frac{3m!}{m!m!m!}$

**Distinction between Permutation and Combination**

Permutation:

Number of arrangements which can be made after selecting few from the available objects. The order of selection does matter.

**Formula for Permutation**

$$n_{Pr} = \frac{n!}{(n-r)!}$$

**Combination:**

Number of ways in which selection of objects from availability would be done. The order of selection does not matter.

**Formula for Combination**

$$n_{Cr} = \frac{n!}{r! \times (n-r)!}$$

Permutation could be termed as Combination in an ordered manner.

**Illustration 89.**

Find the Factorial of 9.

**Solution:**

Factorial of 9:  $9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3,62,880$ .

Therefore, Factorial of 9 = 3,62,880.

**Illustration 90.**

In how many ways 3 rows be selected for vaccination out of 5 rows of college students?

**Solution:**

For selecting 3 rows out of 5 rows of college students for vaccination, the following procedure is considered.

Assuming the selection of rows be done are by one, it can be said that the first row be selected in 5 ways.

So row there remains 4 rows of which the second can be selected in 4 ways. Similarly the third can be selected in 3 ways.

Hence total number of ways of selection of 3 rows of students out of 5 rows is  $5 \times 4 \times 3 = 60$

Thus, 60 ways are there for selecting 3 rows for vaccination out of 5 rows of college students.

**Illustration 91.**

Solve for  $n$ :  $n_{P6} = 13_{P6}$

**Solution:**

Since,  $r = 6$  in the permutations of both sides of the equation,  $n = 13$ .

**Illustration 92.**

If  $n+1_{P3} : n_{P2} = 27 : 9$ , find 'n'.

**Solution:**

$$n+1_{P3} : n_{P2} = 27 : 9$$

$$\left[ \frac{(n+1)!}{(n+1-3)!} \right] : \left[ \frac{n!}{(n-2)!} \right] = 27 : 9$$

$$\text{Or, } \left[ \frac{(n+1)!}{(n-2)!} \right] : \left[ \frac{n!}{(n-2)!} \right] = 27 : 9$$

$$\text{Or, } \frac{(n+1)!}{n!} = \frac{27}{9} \quad \text{Or, } \frac{(n+1)(n!)}{n!} = \frac{27}{9}$$

$$\text{Or, } (n + 1) = 3$$

$$\text{Or, } n = 3 - 1 = 2.$$

**Illustration 93.**

If  $8 \times n - 1P_3 = n - 1P_4$ , find  $n$

**Solution:**

$$8 \times n - 1P_3 = n - 1P_4$$

$$\text{Or, } \frac{n - 1P_3}{n - 1P_4} = \frac{1}{8}$$

$$\text{Or, } \frac{(n-1) \times (n-2) \times (n-3)}{(n-1) \times (n-2) \times (n-3) \times (n-4)} = \frac{1}{8}$$

$$\text{Or, } \frac{1}{(n-4)} = \frac{1}{8}$$

$$\text{Or, } 8 = n - 4$$

$$\text{Or, } n - 4 = 8$$

$$\text{Or, } n = 8 + 4 = 12$$

Therefore,  $n = 12$ .

**Illustration 94.**

If  ${}^n P_5 = 42 \times {}^n P_3$ , find the value of  $n$ .

**Answer.**

$${}^n P_5 = 42 \times {}^n P_3$$

$$\text{Or, } \frac{n!}{(n-5)!} = \frac{n!}{(n-3)!} \times 42$$

$$\text{Or, } \frac{n \times (n-1) \times (n-2) \times (n-3) \times (n-4) \times (n-5)!}{(n-5)!} = \frac{n \times (n-1) \times (n-2) \times (n-3)!}{(n-3)!} \times 42$$

$$\text{Or, } n \times (n-1) \times (n-2) \times (n-3) \times (n-4) = n \times (n-1) \times (n-2) \times 42$$

$$\text{Or, } (n-3) \times (n-4) = 42$$

$$\text{Or, } n^2 - 3n - 4n + 12 = 42$$

$$\text{Or, } n^2 - 7n + 12 - 42 = 0$$

$$\text{Or, } n^2 - 7n - 30 = 0$$

$$\text{Or, } n^2 - 10n + 3n - 30 = 0$$

$$\text{Or, } n(n-10) + 3(n-10) = 0$$

$$\text{Or, } (n-10)(n+3) = 0$$

$$\text{Either } n-10 = 0 \text{ or, } n+3 = 0$$

$$\text{Either } n = 10 \text{ or, } n = -3$$

Since,  $n$  is a positive integer, it should be 10.

Thus, ' $n$ ' = 10.

### Illustration 95.

In how many ways 6 mobiles be gifted to 6 employees?

#### Solution:

The number of employees to be given gifts = 6 employees, the number of mobiles to be gifted = 6 mobiles, so,  $n = 6$  and  $r = 6$ .

Since,  $n = r$ , the number of permutations would be  $n! = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$  ways.

### Illustration 96.

In how many ways 4 laptops be purchased from 7 laptops available at the store?

#### Solution:

In the given question, the number of laptops available in the store - 7 laptops would be ' $n$ ', while the number of laptops to be purchased - 4 laptops would be ' $r$ '.

Number of ways 4 laptops be purchased from 7 laptops would be as follows:

$$\begin{aligned} {}^n P_r &= {}^7 P_4 = \frac{7!}{(7-4)!} \\ &= \frac{7!}{3!} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} \\ &= 7 \times 6 \times 5 \times 4 \\ &= 840 \text{ ways.} \end{aligned}$$

Therefore, 4 laptops can be purchased from 7 laptops available in the store in 840 ways.

### Illustration 97.

In how many ways can 8 articles numbered from 1 to 8 be arranged, if even numbered articles are same?

#### Solution:

Since number of articles to be arranged - 8 articles ( $n = 8$ ) and given that all even numbered articles (Article 2,4,6,8) are same, the no. of ways to arrange such articles would be,

$$\begin{aligned} \frac{8!}{4!} &= \frac{8 \times 7 \times 6 \times 5 \times 4!}{4!} \\ &= 8 \times 7 \times 6 \times 5 = 1680 \text{ ways.} \end{aligned}$$

Thus, 8 articles where all even numbered articles are same, can be arranged in 1680 ways.

### Illustration 98.

Find the number of 5 digit numbers that can be formed with the digits 8, 6,4,3,2,1,5,9.

#### Solution:

The numbers to be formed would be equal to the permutation of 8 different objects taking 5 at a time.

$$\text{Hence, } {}^8P_5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!} = 8 \times 7 \times 6 \times 5 \times 4 = 6,720.$$

Therefore, 6,720 numbers can be formed.

### Illustration 99.

Find the value of  ${}^{10}C_7$ .

#### Solution:

$$\text{Formula: } {}^nC_r = \frac{n!}{r! \times (n-r)!}$$

$${}^{10}C_7 = \frac{10!}{7! \times (10-7)!}$$

$$= \frac{10!}{7! \times 3!}$$

$$= \frac{10 \times 9 \times 8 \times 7!}{7! \times (3 \times 2 \times 1)}$$

$$= \frac{10 \times 9 \times 8}{(3 \times 2 \times 1)}$$

$$= \frac{720}{6} = 120.$$

### Illustration 100.

A bag contains 10 balls. In how many ways can a child take out any 2 balls?

#### Solution:

The number of ways for selection of 2 different balls:-

$${}^{10}C_2 = \frac{10!}{2!(10-2)!}$$

$$= \frac{10!}{2! \cdot 8!} = \frac{10 \times 9 \times 8!}{2! \cdot 8!} = \frac{10 \times 9}{2 \times 1} = \frac{90}{2} = 45$$

### Illustration 101.

Find the value of (i)  $7_{P_5}$ ; (ii)  $7_{P_1}$ ; (iii)  $7_{P_0}$ ; (iv)  $7_{P_7}$ ;

#### Solution:

$$(i) \ 7_{P_5} = \frac{7!}{(7-5)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$$

$$(ii) {}_7P_1 = \frac{7!}{(7-1)!} = \frac{7.6.5.4.3.2.1.}{6.5.4.3.2.1.} = 7$$

$$(iii) {}_7P_0 = \frac{7!}{(7-0)!} = \frac{7.6.5.4.3.2.1.}{7.6.5.4.3.2.1.} = 1$$

$$(iv) {}_7P_7 = \frac{7!}{(7-7)!} = \frac{7.6.5.4.3.2.1.}{1.} = 7.6.5.4.3.2.1 = 5040$$

**Illustration 102.**

If  $n_{P_2} = 110$ , find  $n$

**Solution:**

$$n_{P_2} = 110$$

$$\text{Or, } \frac{n!}{(n-2)!} = 110$$

$$\text{Or, } \frac{n.(n-1).(n-2)!}{(n-2)!} = 110$$

$$\text{Or, } n(n-1) = 110$$

$$\text{Or, } n(n-1) = 11.10$$

$$\text{Or, } n(n-1) = 11(11-1)$$

$$\text{Or, } n = 11$$

**Illustration 103.**

Solve for  $n$ , given  $\frac{n_{P_5}}{n_{P_3}} = 2$

**Solution:**

$$\frac{n_{P_5}}{n_{P_3}} = \frac{\frac{n!}{(n-5)!}}{\frac{n!}{(n-3)!}} = \frac{(n-3)!}{(n-5)!} = (n-3)(n-4) = 2 \quad (\text{Given})$$

$$\text{Or, } n^2 - 7n + 10 = 0$$

$$\text{Or, } (n-5)(n-2) = 0$$

$$\text{Or, } n = 5, 2$$

If  $n = 2$  then both  $n_{P_5}$  and  $n_{P_3}$  have no meaning. So the required value of  $n$  is 5.

**Illustration 104.**

In how many ways 6 books out of 10 different books can be arranged in a book-self so that 3 particular books are always together?

**Solution:**

At first 3 particular books are kept outside. Now remaining 3 books out of remaining 7 books can be arranged in  ${}^7P_3$  ways. In between these three books there are 2 places and at the two ends there are 2 places i.e. total 4 places where 3 particular books can be placed in  ${}^4P_3$  ways. Again 3 particular books can be arranged among themselves in  $3!$  ways.

$$\text{Hence, required no. of ways } {}^7P_3 \times {}^4P_3 \times 3! = \frac{7!}{(7-3)!} \times \frac{4!}{(4-1)!} \times 3 \cdot 2 \cdot 1 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

**Illustration 105.**

In how many ways can the letters of the word TABLE be arranged so that the vowels are always (i) together (ii) separated?

**Solution:**

- (i) In the word TABLE there are 2 vowels and 3 consonants which are all different. Taking the 2 vowels (A, E) as one letter we are to arrange 4 letters (i.e. 3 consonants + 1) which can be done in  $4!$  Ways. Again 2 vowels can be arranged between themselves in  $2!$  ways.

$$\text{Hence, required numbers of ways of arranging the letters so that the vowels are always together} = 4! \times 2! = (4 \cdot 3 \cdot 2 \cdot 1) \times (2 \cdot 1) = 48.$$

- (ii) Required number of ways of arranging the letters so that the vowels are separated =  $5! - 48 = 120 - 48 = 72$   
[Taking all the letters together the no. of possible arrangements =  $5!$ ]

**Illustration 106.**

Find in how many ways can the letters of the word PURPOSE be rearranged –

- (i) Keeping the positions of the vowels fixed;  
(ii) Without changing the relative order of the vowels and consonants.

**Solution:**

- (i) In the word, PURPOSE there are 3 vowels and 4 consonants. Since the positions of all the vowels should remain fixed, we are to rearrange only 4 consonants, of which there are 2Ps. So the arrangement is possible in

$$\frac{4!}{2!} = \frac{4 \times 3 \times 2!}{2!} = 4 \times 3 = 12 \text{ ways}$$

- (ii) The relative order of vowels and consonants should remain unaltered means that vowel will take place of vowel and consonant will take place of consonant. Now the 3 vowels can be arranged among themselves in  $3!$  ways, while 4 consonants with two Ps can be arranged in

$$\frac{4!}{2!} = \frac{4 \times 3 \times 2!}{2!} = 4 \times 3 = 12 \text{ ways}$$

$$\text{So total number of ways of rearrangement in which the given arrangement is included} = (3! \times 12) - 1 = (6 \times 12) - 1 = 72 - 1$$

$$\text{Hence, required number of arrangements} = 72 - 1 = 71$$

**Illustration 107.**

How many numbers between 5000 and 6000 can be formed with the digits 3, 4, 5, 6, 7, 8?

**Solution:**

The number to be formed will be of 4 digits. Further the required numbers are between 5000 and 6000. So the digit 5 is to be placed in the 1st place (from left). Now the remaining 3 places can be filled up by the remaining 5 digits in  $5_{P_3}$  ways.

Hence, required number of nos.  $5_{P_3} = \frac{5!}{2!} = 5.4.3 = 60$

**Illustration 108.**

In how many ways can the letters of the word SUNDAY be arranged? How many of them do not begin with S? How many of them do not begin with S, but end with Y?

**Solution:**

There are 6 different letters in the word SUNDAY, which can be arranged in  $6! = 720$  ways.

Now placing S in first position fixed, the other 5 letters can be arranged in  $(5)! = 120$  ways.

The number of arrangements of letters that do not begin with S =  $(6)! - (5)! = 720 - 120 = 600$

Lastly Placing Y in the last position, we can arrange in  $(5)! = 120$  ways and keeping Y in the last position and S in the first position, we can arrange in  $(4)! = 24$  ways.

Hence, the required no. of arrangements beginning with S but not ending with Y =  $(5)! - 4! = 120 - 24 = 96$ .

**Illustration 109.**

In how many ways 5 boys and 5 girls can take their seats in a round table, so that no two girls will sit side by side.

**Solution:**

If one boy takes his seat anywhere in a round table, then remaining 4 boys can take seats in  $4! = 24$  ways. In each of these 24 ways, between 5 boys, if 5 girls take their seat then no two girls will be beside by side. So in this way 5 girls may be placed in 5 places in  $5! = 120$  ways.

Again the first boy taking seat, may take any one of the 10 seats. i.e. he may take his seat in 10 ways.

Hence, required number ways =  $24 \times 120 \times 10 = 28800$

**Illustration 110.**

In how many ways can the college football team of 11 players be selected from 16 players?

**Solution:**

The required number of ways =  ${}^{16}C_{11} = \frac{16!}{(16-11)!11!} = \frac{16.15.14.13.12}{5.4.3.2} = 4368$

**Illustration 111.**

From a group of 15 men, how many selections of 9 men can be made so as to exclude 3 particular men?

**Solution:**

Excluding 3 particular men in each case, we are to select 9 men out of  $(15-3)$  men. Hence the number of selection is

equal to the number of combination of 12 men taken 9 at a time which is equal to  ${}^{12}C_9 = \frac{12!}{(12-9)!9!} = \frac{12.11.10}{3.2.1} = 220$

**Illustration 112.**

There are seven candidates for a post. In how many ways can a selection of four be made amongst them, so that:

1. 2 persons whose qualifications are below par are excluded?
2. 2 persons with good qualifications are included?

**Solution:**

1. Excluding 2 persons, we are to select 4 out of  $5(=7-2)$  candidates. Number of possible selections =  ${}^5C_4 = 5$  ways
2. In this case, 2 persons are fixed, and we are to select only  $(4-2) = 2$  persons out of  $(7-2)$ , i.e. 5 candidates. Hence the required number of selection =  ${}^5C_2 = \frac{5.4}{2.1} = 10$

**Committee from more than one group:****Illustration 113.**

In how many ways can a committee of 3 ladies and 4 gentlemen be appointed from a meeting consisting of 8 ladies and 7 gentlemen? What will be the number of ways if Mrs. X refuses to serve in a committee having Mr. Y as a member?

**Solution:**

**1st part:** 3 ladies can be selected from 8 ladies in  ${}^8C_3 = \frac{8.7.6}{3.2.1} = 56$  ways

4 gentlemen can be selected from 7 gentlemen in  ${}^7C_4 = \frac{7.6.5}{3.2.1} = 35$  ways

Now, each way of selecting ladies can be associated with each way of selecting gentlemen. Hence, the required no. of ways =  $56 \times 35 = 1960$ .

**2nd part:** If both Mrs. X and Mr. Y are members of the committee then we are to select 2 ladies and 3 gentlemen from 7 ladies and 6 gentlemen respectively. Now 2 ladies can be selected out of 7 ladies in  ${}^7C_2$  ways, and 3 gentlemen can be selected out of 6 gentlemen in  ${}^6C_3$  ways.

Since each way of selecting gentlemen can be associated with each way of selecting ladies.

Hence, No. of ways =  ${}^7C_2 \times {}^6C_3 = 420$  ways

Hence, the required no. of different committees, not including Mrs. X and Mr. Y =  $1960 - 420 = 1540$ .

**Illustration 114.**

From 7 gentlemen and 4 ladies a committee of 5 is to be formed. In how many ways can this be done to include at least one lady?

**Solution:**

Committee of 5 with at least one lady means the committee can be formed with

- (i) 1 lady and 4 gentlemen or
- (ii) 2 ladies and 3 gentlemen or
- (iii) 3 ladies and 2 gentlemen or
- (iv) 4 ladies and 1 gentleman or

For (i), 1 lady can be selected out of 4 ladies in  ${}^4C_1$  ways and 4 gentlemen can be selected from 7 gentlemen in  ${}^7C_4$  ways. Now each way of selecting lady can be associated with each way of selecting gentlemen. So 1 lady and 4 gentlemen can be selected in  ${}^4C_1 \times {}^7C_4$  ways.

Similarly,

Case (ii) can be selected in  $4_{C_2} \times 7_{C_3}$  ways

Case (iii) can be selected in  $4_{C_3} \times 7_{C_2}$  ways

Case (iv) can be selected in  $4_{C_4} \times 7_{C_1}$  ways.

Hence the total number of selections of the committee of 5 so that, at least one lady is included

$$= 4_{C_1} \times 7_{C_4} + 4_{C_2} \times 7_{C_3} + 4_{C_3} \times 7_{C_2} + 4_{C_4} \times 7_{C_1}$$

$$= 4 \times 35 + 6 \times 35 + 4 \times 21 + 1 \times 7 = 441$$

### Illustration 115.

In how many ways can a boy invite one or more of his 5 friends?

**Solution:**

The number of ways =  $5_{C_1} + 5_{C_2} + 5_{C_3} + 5_{C_4} + 5_{C_5} = 5 + 10 + 10 + 5 + 1 = 31$

### Illustration 116.

In how many ways 15 things can be divided into three groups of 4, 5, 6 things respectively.

**Solution:**

The first group can be selected in  $15_{C_4} = \frac{15.14.13.12.11!}{11!.4.3.2.1} = 1365$  ways:

The second group can be selected in  $(15-4)_{C_5} = 11_{C_5} = \frac{11.10.9.8.7.6!}{6!.5.4.3.2} = 462$  ways

And lastly the third group in  $6_{C_6} = 1$  way

Total number of ways =  $1365 \times 462 = 630630$

### Illustration 117.

In how many ways can the letters of the word SIGNAL be arranged so that the vowels occupy only the odd places?

**Solution:**

Number of letters in the word SIGNAL is 6 and they are all different. So there are 3 odd places namely first, third and fifth.

Vowels are only 2. The number of ways in which 2 vowels can occupy 3 places is  $3_{P_2}$ . For each such way the remaining 4 places are to be occupied by 4 remaining consonants. This can be achieved in  $4!$  different ways.

Hence the required arrangements are possible in  $3_{P_2} \times 4! = 144$  ways

### Illustration 118.

How many triangles can be made by joining the vertices of decagon? How many diagonals will it have?

**Solution:**

A triangle is obtained by joining any three of the 10 vertices of a decagon. The number of triangles can be made by joining the vertices of a decagon  $= {}^{10}C_3 = \frac{10 \times 9 \times 8}{3 \times 2} = 120$

Now a diagonal is obtained by joining any two vertices which are not adjacent

Total number of ways in which two vertices of a decagon can be joined  $= {}^{10}C_2 = \frac{10 \times 9}{2} = 45$

Number of lines obtained by joining two adjacent vertices = Number of sides of decagon = 10

Hence number of diagonals =  $45 - 10 = 35$

**Illustration 119.** There are 10 persons in a party. If one can shake hand with others, how many hand shakes can be made.

**Solution:** No. of hand shakes  $= {}^{10}C_2 = 45$ .

**Illustration 120.** How many diagonals are there in a hexagon.

**Solution:**  ${}^6C_2 - 6 = 9$ .

**Illustration 121.** How many triangles can be made by using 8 points, in which no point lie on a straight line.

**Solution:**  ${}^8C_3 = 56$ .

**Illustration 122.** There are 12 points in a plane of which 5 are collinear. Find no. of triangles.

**Solution:**  ${}^{12}C_3 - {}^5C_3 = 220 - 10 = 210$ .

**Illustration 123.** Find no. of straight line obtained by joining 16 points on a plane, where none of them lie on the same line.

**Solution:**  ${}^{16}C_2 = 120$ .

**Illustration 124.** In how many ways can a team of 11 be chosen from 14 football players, two of them can only be goalkeeper.

**Solution:**  ${}^{12}C_{10} \times {}^2C_1 = 132$ .

**Illustration 125.** A polygon has 27 diagonals. Find no. of its sides.

**Solution:** Let there are n sides of a polygon.

No. of diagonals of a polygon having n sides is  $= \frac{n(n-3)}{2}$

So  $\frac{n(n-3)}{2} = 27$

Or  $n^2 - 3n - 54 = 0$

Or  $(n-9)(n+6) = 0$

Or  $n = 9$ .

# Quadratic Equation (Basic Concepts)

## 2.4

### Definition and Form of Equation:

#### Quadratic Expression:

An expression which is of the form  $ax^2+bx+c$  is called quadratic expression.

In this expression  $a$ ,  $b$  and  $c$  are constants. An important point regarding a quadratic expression is the fact that the maximum power (or index) of the variable  $x$  in the expression is 2. In other words ‘Degree’ of a Quadratic expression is 2.

The term ‘Degree’ refers to the highest power of the variable in the expression. For expressions with single variable, the highest power of the variable is considered to find out the degree and for expressions involving multiple variable the highest sum of powers of different variables in any of the terms of the expression is taken as the degree.

For the expression  $x^3 + 3x^3y^2 + 2xy^3 + 4y^2 - 2y + 1$  the Degree is 5 because sum of the powers of  $x^3y^2$  is  $(3+2)=5$  and it is the highest among all the terms.

Ex:  $2x^2 + 4x + 7$  is a quadratic expression.

Ex:  $(2+i)x^2 - 3ix + 6(2+i)$  is also a quadratic expression.

#### Quadratic Equation:

Any equation expressed in the form  $ax^2 + bx + c = 0$ , is termed as Quadratic Equation.

For Example,

a.  $15x^2 + 28x + 13 = 0$

Here the equation is expressed in the form of  $ax^2 + bx + c = 0$  with  $a = 15$ ,  $b = 28$  and  $c = 13$

b.  $x^2 = 30x - 6$  Or,  $x^2 - 30x + 6 = 0$  Or,  $1x^2 + (-30)x + 6 = 0$

Here also the equation is expressed in the form of  $ax^2 + bx + c = 0$  with  $a = 1$ ,  $b = -30$  and  $c = 6$

#### Roots and Nature of Roots:

Each Quadratic Equation has two solutions called roots and normally denoted as  $\alpha$  and  $\beta$

#### Solution methods of quadratic equation:

1. Factorization method
2. Completing square method:

Find the roots of the quadratic equation  $ax^2 + bx + c = 0$

Given

Quadratic equation  $ax^2 + bx + c = 0$

Multiplying both sides by  $4a$  we get.

$$4a^2x^2 + 4abx + 4ac = 0$$

$$\text{Or, } 4a^2x^2 + 4abx = -4ac$$

Adding  $b^2$  on both sides we get

$$\text{Or, } 4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

$$\text{Or, } (2ax)^2 + 2(2abx) + b^2 = b^2 - 4ac$$

$$\text{Or, } (2ax + b)^2 = b^2 - 4ac$$

$$\text{Or, } (2ax + b) = \pm\sqrt{b^2 - 4ac}$$

$$\text{Or, } 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\text{Or, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ This formula is also called Sridhar Acharya's formula}$$

So the roots  $\alpha$  and  $\beta$  of the quadratic equation are given as

$$\text{Value of } \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Value of } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Here  $b^2 - 4ac$  is called the Discriminant and it describes the nature of the roots of the quadratic equation and is denoted by  $\Delta = b^2 - 4ac$

**Discriminant of a Quadratic Equation:**  $b^2 - 4ac$

**Nature of Roots:**

1. If  $b^2 - 4ac > 0$  then the roots are real and unequal.
2. If  $b^2 - 4ac = 0$  then the roots are real and equal.
3. If  $b^2 - 4ac < 0$  then the roots are imaginary or conjugate complex numbers.
4. If  $b^2 - 4ac > 0$  and perfect square then the roots are rational and unequal.
5. If  $b^2 - 4ac > 0$  but not a perfect square then the roots are irrational numbers.
6. If one root of a quadratic equation  $ax^2 + bx + c = 0$ , is  $a + ib$  then the other root will be  $a - ib$

7. If one root of quadratic equation  $ax^2 + bx + c = 0$  is  $a + \sqrt{d}$  then the other root is  $a - \sqrt{d}$ .

### To Form a Quadratic Equation

For forming a Quadratic Equation, the following formula needs to be followed:

#### Formula:

$$x^2 - (\text{Sum of the Roots}) \times x + (\text{Product of the Roots}) = 0$$

Sum of the Roots:  $\alpha + \beta$

Product of the Roots :  $\alpha\beta$

**Case I:** Sum of the roots

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$$

**Case II:** Product of the roots

$$\alpha\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \left(\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{4ac}{4a^2} = \frac{c}{a}$$

#### Special cases:

(i) If the constant term vanishes then one of the root is zero.

Explanation:

$$\text{If } c = 0, \text{ then } \alpha\beta = \frac{c}{a} = \frac{0}{a} = 0$$

i.e. either  $\alpha = 0$  or  $\beta = 0$  or both  $\alpha$  &  $\beta = 0$

(ii) If the coefficient of  $x$  vanishes then the roots are equal in magnitude and opposite in sign.

Explanation:

$$\text{If } b = 0, \alpha + \beta = -\frac{b}{a} = -\frac{0}{a} = 0$$

i.e. either  $\alpha = -\beta$  or  $\beta = -\alpha$

#### Methods for Solving Quadratic Equation:

1. Factorisation
2. Using Square Roots
3. Using Quadratic Formula or Sridhar Acharya's Formula
1. Factorisation

The Equation is solved by factorising the quadratic expression involved with the equation.

For example:

Equation:  $x^2 - 5x + 6 = 0$

**Solution:**

$x^2 - 5x + 6 = 0$

Or,  $x^2 - 2x - 3x + 6 = 0$

Or,  $x(x-2) - (x-2) = 0$

Or,  $(x-2)(x-3) = 0$

Or,  $x = 2$  and  $x = 3$

2. Using Square Roots

The value of  $x$  would be a square number when solving the equation.

3. Using Quadratic Formula or Sridhar Acharya’s Formula

Formula for Solving Quadratic Equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Illustration 126.**

Solve  $\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1$ ;

**Solution:**

$$\left(\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7}\right) = 1 \dots \dots \dots (1)$$

Now  $(5x^2 - 6x + 8) - (5x^2 - 6x - 7) = 15$

Or,  $\frac{(5x^2 - 6x + 8) - (5x^2 - 6x - 7)}{\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7}} = \frac{15}{1}$  [Substituting  $\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1$  from equation (1)]

$$\text{Or, } \frac{\left\{\left(\sqrt{5x^2 - 6x + 8}\right)^2 - \left(\sqrt{5x^2 - 6x - 7}\right)^2\right\}}{\left\{\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7}\right\}} = \frac{15}{1}$$

$$\text{Or, } \frac{\left(\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7}\right)\left(\sqrt{5x^2 - 6x + 8} + \sqrt{5x^2 - 6x - 7}\right)}{\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7}} = \frac{15}{1}$$

Or,  $\left(\sqrt{5x^2 - 6x + 8} + \sqrt{5x^2 - 6x - 7}\right) = 15 \dots \dots \dots (2)$

By (1) + (2) we get  $2\sqrt{5x^2 - 6x + 8} = 16$

$$\text{Or, } \sqrt{5x^2 - 6x + 8} = 8$$

$$\text{Or, } 5x^2 - 6x + 8 = 64$$

$$\text{Or, } 5x^2 - 6x - 56 = 0$$

$$\text{Or, } x = \frac{6 \pm \sqrt{36 + 4 \times 5 \times 56}}{2 \times 5}$$

$$\text{Or, } x = \frac{6 + \sqrt{36 + 4 \times 5 \times 56}}{2 \times 5} \text{ and } x = \frac{6 - \sqrt{36 + 4 \times 5 \times 56}}{2 \times 5}$$

$$\text{Or, } x = 4 \text{ and } x = -2.8$$

### Illustration 127.

Solve  $x^{(2/3)} + x^{(1/3)} = 2$ ;

**Solution:**

Put  $x^{1/3} = t$  we get

$$t^2 + t - 2 = 0$$

$$\text{Or, } t^2 + 2t - t - 2 = 0 \text{ Or, } t(t + 2) - 1(t + 2) = 0$$

$$\text{Or, } (t + 2)(t - 1) = 0$$

$$\text{Or, } t = -2 \text{ and } 1$$

$$\text{Or, } x^{1/3} = -2 \text{ and } 1$$

$$\text{Or, } x = -8 \text{ and } 1$$

### Illustration 128.

Evaluate  $\sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots \infty}}}}$

**Solution:**

$$\text{Let } x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots \infty}}}}$$

$$\text{Or, } x^2 = 6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots \infty}}}$$

$$\text{Or, } x^2 = 6 + x$$

$$\text{Or, } x^2 - x - 6 = 0$$

$$\text{Or, } (x - 3)(x + 2) = 0$$

$$\text{Or, } x = 3 \text{ and } -2$$

**Illustration 129.**

Evaluate  $2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots \dots \dots \infty}}}$ ;

**Solution:**

Let  $x = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots \dots \dots \infty}}}$

Or,  $x = 2 + \frac{1}{x}$

Or,  $x^2 = 2x + 1$

Or,  $x^2 - 2x - 1 = 0$

Or,  $x = \frac{2 + \sqrt{4+4}}{2}$  and  $\frac{2 - \sqrt{4+4}}{2}$

Or,  $x = 1 + \sqrt{2}$  and  $1 - \sqrt{2}$

**Illustration 130.**

Solve  $\frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = \frac{a}{x}, x \neq 0$

**Solution:**

Rationalising the denominator of L.H.S we get

$\frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} \times \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}} = \frac{a}{x}, x \neq 0$

Or,  $\frac{(\sqrt{a+x} + \sqrt{a-x})^2}{(a+x) - (a-x)} = \frac{a}{x}$

Or,  $\frac{a+x+a-x+2\sqrt{a+x} \times \sqrt{a-x}}{2x} = \frac{a}{x}$

Or,  $\frac{2a+2\sqrt{a+x} \times \sqrt{a-x}}{2x} = \frac{a}{x}$

Or,  $a + \sqrt{a^2 - x^2} = a$

Or,  $x^2 = a^2$

Or,  $x = a$  and  $-a$

**Illustration 131.**

$$\text{Solve } (x^2 + 2)^2 + 8x^2 = 6x(x^2 + 2);$$

**Solution:**

$$\text{Let } x^2 + 2 = A$$

$$\text{Or, } A^2 + 8x^2 - 6Ax = 0$$

$$\text{Or, } 8x^2 - 6Ax + A^2 = 0$$

$$\text{Or, } 8x^2 - 4Ax - 2Ax + A^2 = 0$$

$$\text{Or, } 4x(2x - A) - A(2x - A) = 0$$

$$\text{Or, } (2x - A)(4x - A) = 0$$

$$\text{Or, } x = \frac{A}{2} \text{ and } \frac{A}{4}$$

$$\text{Or, } x^2 - 2x + 2 = 0 \text{ and } x^2 - 4x + 2 = 0$$

$$\text{Or, } x = \frac{2 \pm \sqrt{4-8}}{2} \text{ and } \frac{4 \pm \sqrt{16-8}}{2}$$

$$\text{Or, } x = 1 \pm i \text{ and } 2 \pm \sqrt{2}$$

**Illustration 132.**

A trader bought a number of articles for ₹1,200. Ten articles were damaged and he sold each of the rest at ₹2 more than what he paid for, thus clearing a profit of ₹60 on the whole transaction. Taking the number of articles he bought as  $x$  form an equation in  $x$  and solve it.

**Solution:**

Let  $x$  be the number of articles purchased at ₹1200

$$\text{So purchase price per unit} = \frac{1200}{x}$$

$$\text{Therefore sale price per unit} = \frac{1200}{x} + 2$$

Since 10 articles were damaged so no of articles sold =  $x - 10$

Since there is a profit of ₹60, then total sale proceeds will be =  $1200 + 60 = ₹1260$

$$\text{Therefore we can write } (x - 10) \times \left( \frac{1200}{x} + 2 \right) = 1260$$

$$\text{Or, } (x - 10)(1200 + 2x) = 1260x$$

$$\text{Or, } 1200x - 12000 + 2x^2 - 20x = 1260x$$

$$\text{Or, } 2x^2 - 80x - 12000 = 0$$

$$\text{Or, } x^2 - 40x - 6000 = 0$$

$$\text{Or, } x = \frac{40 \pm \sqrt{1600 + 4 \times 6000}}{2}$$

Or,  $x = 100$  and  $-60$

$x$  being physical quantity can never be negative. So  $x = 100$  units

### Illustration 133.

Solve  $x(x+2)(x^2-1) = -1$ ;

**Solution:**

$$x(x+2)(x^2-1) = x(x+2)(x+1)(x-1) = -1$$

$$\text{Or, } x(x+1)(x+2)(x-1) = -1$$

$$\text{Or, } (x^2+x)(x^2+x-2) = -1$$

$$\text{Let } x^2+x = A$$

$$\text{So } A(A-2) = -1$$

$$\text{Or, } A^2 - 2A + 1 = 0$$

$$\text{Or, } (A-1)^2 = 0$$

$$\text{Or, } A = 1$$

$$\text{Or, } x^2 + x - 1 = 0$$

$$\text{Or, } x = \frac{-1 \pm \sqrt{5}}{2}$$

### Illustration 134.

If  $\alpha$  and  $\beta$  are the roots of the equation  $x^2 - 4x + 2 = 0$ , find the value of  $\alpha^2 + \beta^2$ ;

**Solution:**

As  $\alpha$  and  $\beta$  are the roots of the equation  $x^2 - 4x + 2 = 0$  which is of the form  $ax^2 + bx + c = 0$  with  $a = 1$ ,  $b = -4$  and  $c = 2$ , we can write

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-4)}{1} = 4 \text{ and } \alpha\beta = \frac{c}{a} = \frac{2}{1} = 2$$

$$\text{Therefore } \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (4)^2 - 2 \times 2 = 16 - 4 = 12$$

### Illustration 135.

If the roots of the equation  $x^2 - lx + m = 0$  differ by 1, then prove that  $l^2 = 4m + 1$ ;

**Solution:**

Given equation  $x^2 - lx + m = 0$  is of the form  $ax^2 + bx + c = 0$  with  $a = 1$ ,  $b = -l$  and  $c = m$

Let the roots be  $\alpha$  &  $\alpha + 1$

$$\text{So } \alpha + \alpha + 1 = \frac{-b}{a} = l \text{ \& } \alpha(\alpha + 1) = \frac{c}{a} = m$$

$$\text{So } 2\alpha = l-1$$

$$\text{Or, } \alpha = \frac{l-1}{2} \text{ \& putting this value in } \alpha(\alpha+1) = m$$

$$\text{Or, } \frac{l-1}{2} \left( \frac{l-1}{2} + 1 \right) = m$$

$$\text{Or, } \frac{l-1}{2} \times \frac{l+1}{2} = m$$

$$\text{Or, } l^2 - 1 \equiv 4m$$

$$\text{Or, } l^2 = 4m + 1 \text{ (Proved)}$$

### Illustration 136.

If the equation  $x^2 - ax + b = 0$  and  $x^2 - cx + d = 0$  have one root in common and the second equation has equal roots, then prove that  $ac = 2(b + d)$  ;

#### Solution:

The equation  $x^2 - cx + d = 0$  is of the form  $ax^2 + bx + c = 0$  with  $a = 1$ ,  $b = -c$  and  $c = d$

Since  $x^2 - cx + d = 0$  has equal roots we can write

$$\text{Discriminant} = 0 \text{ i.e. } b^2 - 4ac = 0$$

$$c^2 - 4d = 0$$

$$\text{So } d = \frac{c^2}{4} \quad \dots (1)$$

Putting this value in  $x^2 - cx + d = 0$  we get

$$x^2 - cx + \frac{c^2}{4} = 0$$

$$\text{Or, } 4x^2 - 4cx + c^2 = 0$$

$$\text{Or, } (2x - c)^2 = 0$$

$$\text{Or, } x = \frac{c}{2}$$

Since equations  $x^2 - ax + b = 0$  and  $x^2 - cx + d = 0$  have one root in common, we can write

$$\left( \frac{c}{2} \right)^2 - a \frac{c}{2} + b = 0$$

$$\text{Or, } \frac{c^2}{4} - a \frac{c}{2} + b = 0$$

$$\text{Or, } d - a \frac{c}{2} + b = 0 \quad [ \because \text{From (1) } d = \frac{c^2}{4} ]$$

$$\text{Or, } ac = 2(b + d) \text{ (Proved)}$$

### Illustration 137.

If the roots of the equation  $2x^2 + (k+1)x + (k^2 - 5k + 6) = 0$  are of opposite sign then show that  $2 < k < 3$ ;

**Solution:**

Given equation  $2x^2 + (k+1)x + (k^2 - 5k + 6) = 0$  is of the form  $ax^2 + bx + c = 0$  with  $a = 2$ ,  $b = k + 1$  and  $c = k^2 - 5k + 6$

Since the roots are of opposite sign as well as roots are real and distinct

So discriminant  $> 0$  and product of roots  $< 0$

$$\text{Or, } b^2 - 4ac > 0 \text{ and } \frac{c}{a} < 0$$

$$\text{Or, } (k+1)^2 - 4 \times 2 \times (k^2 - 5k + 6) > 0 \text{ and } \frac{k^2 - 5k + 6}{2} < 0$$

First condition is always true when second holds

$$\text{Hence } \frac{k^2 - 5k + 6}{2} < 0$$

$$\text{Or } (k-3)(k-2) < 0$$

This is possible only if  $2 < k < 3$  (Proved)

**Illustration 138.**

If the equations  $3x^2 + px + 1 = 0$  and  $2x^2 + qx + 1 = 0$  have a common root, show that  $2p^2 + 3q^2 - 5pq + 1 = 0$ ;

**Solution:**

Let  $\alpha$  be the common root of the equations given. So we can write

$$3\alpha^2 + p\alpha + 1 = 0 \dots \dots \dots (1) \ \&$$

$$2\alpha^2 + q\alpha + 1 = 0 \dots \dots \dots (2)$$

By (1)  $\times$  2 and (2)  $\times$  3 we get

$$6\alpha^2 + 2p\alpha + 2 = 0 \dots \dots \dots (3) \ \&$$

$$6\alpha^2 + 3q\alpha + 3 = 0 \dots \dots \dots (4)$$

By (4)  $-$  (3) we get

$$\alpha(3q - 2p) = -1$$

$$\text{So } \alpha = -\frac{1}{3q - 2p}$$

Putting this value in (1)

$$3 \times \frac{1}{(3q - 2p)^2} - \frac{p}{3q - 2p} + 1 = 0$$

$$\text{Or, } 3 - p(3q - 2p) + (3q - 2p)^2 = 0$$

$$\text{Or, } 3 - 3pq + 2p^2 + 9q^2 - 12pq + 4p^2 = 0$$

$$\text{Or, } 6p^2 + 9q^2 - 15pq + 3 = 0$$

$$\text{Or, } 2p^2 + 3q^2 - 5pq + 1 = 0 \text{ (Proved)}$$

### Illustration 139.

If  $\alpha$  &  $\beta$  be the roots of the equation  $2x^2 - 3x + 1 = 0$  find an equation whose roots are  $\frac{\alpha}{2\beta+3}$  &  $\frac{\beta}{2\alpha+3}$

#### Solution:

Given equation  $2x^2 - 3x + 1 = 0$  is of the form  $ax^2 + bx + c = 0$  with  $a = 2$ ,  $b = -3$  and  $c = 1$

$$\alpha + \beta = \frac{-b}{a} = \frac{3}{2}, \quad \alpha\beta = \frac{c}{a} = \frac{1}{2}$$

The required equation is  $x^2 - (\text{sum of the roots})x + \text{product of the roots} = 0$

$$\text{Sum of the roots} = \frac{\alpha}{2\beta+3} + \frac{\beta}{2\alpha+3} = \frac{2\alpha^2 + 3\alpha + 2\beta^2 + 3\beta}{(2\beta+3)(2\alpha+3)} = \frac{2(\alpha^2 + \beta^2) + 3(\alpha + \beta)}{(4\alpha\beta + 6\alpha + 6\beta + 9)} = \frac{2(\alpha + \beta)^2 - 4\alpha\beta + 3(\alpha + \beta)}{\{4\alpha\beta + 6(\alpha + \beta) + 9\}}$$

$$\text{Or, Sum of the roots} = \frac{2(\alpha + \beta)^2 - 4\alpha\beta + 3(\alpha + \beta)}{\{4\alpha\beta + 6(\alpha + \beta) + 9\}} = \frac{2 \times \left(\frac{9}{4}\right) - 4\left(\frac{1}{2}\right) + 3\left(\frac{3}{2}\right)}{\left(4 \times \frac{1}{2} + 6 \times \frac{3}{2} + 9\right)} = \frac{4.5 - 2 + 4.5}{2 + 9 + 9} = \frac{7}{20}$$

$$\text{Product of the roots} = \frac{\alpha}{2\beta+3} \times \frac{\beta}{2\alpha+3} = \frac{\alpha\beta}{(4\alpha\beta + 6\alpha + 6\beta + 9)} = \frac{\alpha\beta}{4\alpha\beta + 6(\alpha + \beta) + 9} = \frac{\frac{1}{2}}{4 \cdot \frac{1}{2} + 6 \cdot \frac{3}{2} + 9} = \frac{\frac{1}{2}}{20} = \frac{1}{40}$$

$$\text{So the required equation is } x^2 - \frac{7}{20}x + \frac{1}{40} = 0$$

$$\text{Or, } 40x^2 - 14x + 1 = 0$$

### Illustration 140.

$$\text{Solve } \sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = 2\frac{1}{6}$$

#### Solution:

$$\text{Put } \frac{x}{1-x} = t^2$$

$$\text{We get } t + \frac{1}{t} = \frac{13}{6}$$

$$\text{Or } 6t^2 + 6 = 13t$$

$$\text{Or } 6t^2 - 13t + 6 = 0$$

$$\text{Or } 6t^2 - 4t - 9t + 6 = 0$$

$$\text{Or } 2t(3t-2) - 3(3t-2) = 0$$

$$\text{Or } (2t-3)(3t-2) = 0$$

$$\text{Or } t = \frac{3}{2} \text{ and } \frac{2}{3}$$

$$\text{Now } t = \frac{3}{2} \text{ Or, } \frac{x}{1-x} = \frac{9}{4} \text{ (Substituting } t = \sqrt{\frac{x}{1-x}} \text{). So } 4x = 9 - 9x$$

$$\text{Or } 13x = 9. \text{ So } x = \frac{9}{13}$$

Also

$$t = \frac{2}{3} \text{ Or, } \frac{x}{1-x} = \frac{4}{9}$$

$$\text{Or } 9x = 4 - 4x$$

$$\text{Or } 13x = 4. \text{ So } x = \frac{4}{13}$$

$$\text{Required solution is } x = \frac{9}{13} \text{ and } \frac{4}{13}$$

#### Illustration 141.

For  $16a^2 + 16a + 1 = 0$ , prove that the roots are real

**Solution:**

$$16a^2 + 16a + 1 = 0 \equiv ax^2 + bx + c = 0$$

$$\text{So, } a = 16, b = 16, c = 1$$

Formula for finding the value of Discriminant:  $b^2 - 4ac$

$$= (16)^2 - 4 \times (16) \times (1)$$

$$= 256 - 64$$

$$= 192 > 0$$

Since,  $b^2 - 4ac > 0$ , roots of the equation  $16a^2 + 16a + 1 = 0$  are Real. (Proved)

#### Illustration 142.

For  $12a^2 + 10a + 2 = 0$  prove that the roots are rational and real

**Solution:**

$$12a^2 + 10a + 2 = 0 \equiv ax^2 + bx + c = 0$$

$$\text{So, } a = 12, b = 10, c = 2$$

Formula for finding the value of Discriminant:  $b^2 - 4ac$

$$(10)^2 - 4 \times (12) \times (2) = 100 - 96 = 4 > 0$$

Since,  $b^2 - 4ac > 0$ , roots of the equation  $12a^2 + 10a + 2 = 0$  are real. Since,  $b^2 - 4ac$  is a perfect square ( $2^2 = 4$ ), roots are rational and real. (Proved)

### Illustration 143.

For  $32a^2 + 16a + 2 = 0$  prove that the roots are real & equal

#### Solution:

$$32a^2 + 16a + 2 = 0 \equiv ax^2 + bx + c = 0$$

$$\text{So, } a = 32, b = 16, c = 2$$

Formula for finding the value of Discriminant:  $b^2 - 4ac$

$$= (16)^2 - 4 \times (32) \times (2)$$

$$= 256 - 256$$

$$= 0$$

Since,  $b^2 - 4ac = 0$ , roots of the equation  $16a^2 + 16a + 1 = 0$  are Real and Equal.

### Illustration 144.

If sum of roots is -7 and product of roots is -14, form a quadratic equation.

#### Solution:

For forming a Quadratic Equation, the following formula needs to be used:

Formula:

$$x^2 - (\text{Sum of the Roots}) \times x + (\text{Product of the Roots}) = 0$$

$$\text{Sum of the Roots: } \alpha + \beta = -7$$

$$\text{Product of the Roots: } \alpha \times \beta = -14$$

$$\text{Required equation is } x^2 - (-7) \times x + (-14) = 0$$

$$\text{Or, } x^2 + 7x - 14 = 0$$

Quadratic Equation is  $x^2 + 7x - 14 = 0$  when sum of the roots is -7 and product of the roots is -14.

### Illustration 145.

Identify whether the equation  $25d^2 + 36e^2 + 5 = 1$  is Quadratic Equation or not.

#### Solution:

No, the given equation is not a polynomial equation with degree 2 for same variable. The given equation has 2 different variables. Hence, it is not a Quadratic Equation.

### Illustration 146.

Determine the nature of roots for the equation:  $10y^2 + 11y + 3 = 0$

#### Solution:

$$\text{Quadratic equation: } 10y^2 + 11y + 3 = 0 \equiv ax^2 + bx + c = 0$$

So,  $a = 10, b = 11, c = 3$

Formula for Discriminant (D):  $b^2 - 4ac$

$$= (11)^2 - 4 \times (10) \times (3)$$

$$= 121 - 120 = 1 > 0$$

Since, value of discriminant (D)  $> 0$ , the roots are Real and also D is a perfect square, the roots are rational. Hence, the roots of the given equation are real and rational.

#### Illustration 147.

Determine the nature of roots for the equation:  $3y^2 - 5y + 3 = 0$

**Solution:**

Quadratic equation:  $3y^2 + (-5)y + 3 = 0 \equiv ax^2 + bx + c = 0$

So,  $a = 3, b = -5, c = 3$

Formula for Discriminant (D):  $b^2 - 4ac$

$$= (-5)^2 - 4 \times (3) \times (3)$$

$$= 25 - 36 = -11 < 0$$

Since, value of the discriminant (D)  $< 0$ , the roots are Imaginary.

#### Illustration 148.

When two roots are equal and of the value 5 then form the quadratic equation.

**Solution:**

Let the two roots be  $\alpha$  and  $\beta$ ,

Given condition, both roots are equal and the value is 5,

To form a Quadratic Equation, the Formula is:  $x^2 - (\text{Sum of the Roots}) \times x + (\text{Product of the Roots}) = 0$

$$\text{Sum of Roots} = \alpha + \beta = 5 + 5 = 10$$

$$\text{Product of Roots: } \alpha \times \beta = 5 \times 5 = 25$$

Required Quadratic Equation is:  $x^2 - (10) \times x + (25) = 0$

$$\text{Or, } x^2 - 10x + 25 = 0.$$

#### Illustration 149.

If the roots of a quadratic equation are reciprocal and one root be  $1/2$  then form the quadratic equation.

**Solution:**

Let the two roots be  $\alpha$  and  $\beta$ ,

Given condition, One root is reciprocal of the other one and the value is  $1/2$ ,

Value of other root would be 2.

To form a Quadratic Equation the Formula is:  $x^2 - (\text{Sum of the Roots}) \times x + (\text{Product of the Roots}) = 0$

$$\text{Sum of the Roots: } \alpha + \beta = 1/2 + 2 = 5/2$$

Product of the Roots:  $\alpha \times \beta = 1/2 \times 2 = 1$

Required Quadratic Equation is:  $x^2 - (5/2) \times x + (1) = 0$  Or,  $2x^2 - 5x + 2 = 0$ .

### Illustration 150.

Solve using factorisation method:  $y^2 - 14y + 49 = 0$

#### Solution:

Given quadratic equation:  $y^2 - 14y + 49 = 0$

$$\text{or, } y^2 - 7y - 7y + 49 = 0$$

$$\text{or, } y(y - 7) - 7(y - 7) = 0$$

$$\text{or, } (y - 7)(y - 7) = 0$$

$$(y - 7) = 0 \text{ and } (y - 7) = 0$$

$$y = 7 \text{ and } y = 7.$$

Hence, the value of  $y$  in  $y^2 - 14y + 49 = 0$  would be 7.

### Illustration 151.

Solve using square roots method:  $y^2 - 10y + 25 = 0$

#### Solution:

Given quadratic equation:  $y^2 - 10y + 25 = 0$

$$\text{or, } y^2 - 2 \times 5y + (5)^2 \equiv (y - 5)^2 = 0$$

$$\text{or, } (y - 5) = 0$$

$$\text{or, } y = 5.$$

Hence, the value of  $y$  in  $y^2 - 10y + 25 = 0$  would be 5.

### Illustration 152.

Solve:  $11x^2 + 12x - 5 = 0$ .

#### Solution:

Given Quadratic Equation:  $11x^2 + 12x - 5 = 0$  cannot be solved using either factorisation method or square roots method, it would be solved using following formula.

Formula for Solving Quadratic Equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Given Equation:  $11x^2 + 12x - 5 = 0$  is of the form  $ax^2 + bx + c = 0$  with  $a = 11$ ,  $b = 12$ ,  $c = -5$

Replacing values of  $a$ ,  $b$  and  $c$  in formula:

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4 \times 11 \times (-5)}}{2 \times 11}$$

$$\text{Or, } x = \frac{-12 \pm \sqrt{144 + 220}}{22}$$

$$\text{Or, } x = \frac{-12 \pm \sqrt{364}}{22}$$

$$\text{Or, } x = \frac{-12 \pm 19.08}{22}$$

$$\text{Or, } x = \frac{-12 + 19.08}{22} \text{ and } x = \frac{-12 - 19.08}{22}$$

$$\text{Or, } x = \frac{7.08}{22} \text{ and } x = \frac{-31.08}{22}$$

$$\text{Or, } x = 0.32 \text{ and } x = -1.41$$

Therefore, the values of  $x$  for  $11x^2 + 12x - 5 = 0$  are 0.32 and  $-1.41$ .

#### Illustration 153.

$$\text{Solve } x^2 - 7x + 12 = 0$$

**Solution:**

Given equation can be expressed as  $x^2 - 3x - 4x + 12 = 0$  or,  $x(x - 3) - 4(x - 3) = 0$

$$\text{Or, } (x - 3)(x - 4) = 0 \text{ Or, } x = 3, 4$$

Alternatively from the given equation we can write  $a = 1$ ,  $b = -7$  &  $c = 12$

$$\text{So } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 1 \cdot 12}}{2 \cdot 1} = \frac{7 \pm \sqrt{49 - 48}}{2}$$

$$\text{Or, } x = \frac{7 + \sqrt{49 - 48}}{2} \text{ and } \frac{7 - \sqrt{49 - 48}}{2} \text{ i.e. } x = 4 \text{ and } 3$$

#### Illustration 154.

$$\text{Solve } x^4 - 10x^2 + 9 = 0$$

**Solution:**

Taking  $x^2 = u$  we get the transformed form of the given equation as  $u^2 - 10u + 9 = 0$

$$\text{Or, } (u - 9)(u - 1) = 0. \text{ So } u = 9 \text{ and } u = 1$$

$$\text{Or, } u = x^2 = 9. \text{ So } x = \pm 3. \text{ \& } u = x^2 = 1, \text{ So } x = \pm 1.$$

Here the power of  $x$  is 4, so we get four values of  $x$ .

#### Illustration 155.

$$\text{Solve } (x+1)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}} = 2^{\frac{1}{3}}$$

**Solution:**

Cubing both sides we get  $(1+x) + (1-x) + 3(x+1)^{1/3} \times (1-x)^{1/3} \left[ (x+1)^{1/3} + (1-x)^{1/3} \right] = 2$

Or,  $2 + 3(x+1)^{1/3} \times (1-x)^{1/3} 2^{1/3} = 2$  [Since  $(x+1)^{1/3} + (1-x)^{1/3} = 2^{1/3}$ ]

Or,  $3(1-x^2)^{1/3} \cdot 2^{1/3} = 0$

Or,  $(1-x^2)^{1/3} = 0$ , or  $x^2 = 1$ , Or  $x = \pm 1$

### Illustration 156.

Solve  $\frac{6-x}{x^2-4} = \frac{x}{x+2} + 2$

**Solution:**

$$\frac{6-x}{x^2-4} = \frac{x}{x+2} + 2$$

$$\text{Or, } (6-x) = \frac{x}{x+2} \times (x^2-4) + 2(x^2-4)$$

$$\text{Or, } 6-x = x(x-2) + 2(x^2-4)$$

$$\text{Or, } 6-x = x^2 - 2x + 2x^2 - 8$$

$$\text{Or, } 6-x = 3x^2 - 2x - 8$$

$$\text{Or, } 3x^2 - x - 14 = 0$$

Drawing similarity of the above equation with  $ax^2 + bx + c = 0$  we can write  $a = 3$ ,  $b = -1$  &  $c = -14$  here.

$$\text{So } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-14)}}{2 \cdot 3} = \frac{1 \pm \sqrt{1+168}}{6}$$

$$\text{Or, } x = \frac{1 + \sqrt{169}}{6} \text{ and } \frac{1 - \sqrt{169}}{6} \text{ i.e. } x = 7/3 \text{ and } -2$$

$$\text{Alternatively : } 3x^2 - x - 14 = 0, \text{ Or, } 3x^2 - 7x + 6x - 14 = 0$$

$$\text{Or, } 3x^2 + 6x - 7x - 14 = 0 \text{ Or, } 3x(x+2) - 7(x+2) = 0$$

$$\text{Or, } (x+2)(3x-7) = 0 \text{ Or, } x = -2 \text{ and } 7/3$$

Now  $x = -2$  yields undefined result when substituted in the given equation. Thus  $x \neq -2$ .

So the required value of  $x$  is  $7/3$

### Illustration 157.

Solve  $4^x - 3 \cdot 2^{x+2} + 2^5 = 0$

**Solution:**

$$4^x - 3 \cdot 2^{x+2} + 2^5 = 0$$

$$\text{Or, } 2^{2x} - 3 \cdot 2^x 2^2 + 32 = 0 \text{ Or, } (2^x)^2 - 12 \cdot 2^x + 32 = 0$$

So Let  $2^x = z$ , we get  $z^2 - 12z + 32 = 0$

$$\text{Or, } z^2 - 4z - 8z + 32 = 0$$

$$\text{Or, } z(z - 4) - 8(z - 4) = 0$$

$$\text{Or, } (z - 4)(z - 8) = 0$$

$$\text{Or, } z = 4 \text{ and } z = 8$$

$$\text{Or, } z = 2^x = 4 = 2^2 \text{ and } z = 2^x = 8 = 2^3$$

$$\text{Or, } x = 2 \text{ and } x = 3$$

**Illustration 158.**

$$\text{Solve } x^2 + 7x + \sqrt{x^2 + 7x + 9} = 3$$

**Solution:**

$$x^2 + 7x + \sqrt{x^2 + 7x + 9} = 3$$

$$\text{Adding 9 to both sides, we have } x^2 + 7x + 9 + \sqrt{x^2 + 7x + 9} = 12$$

Now substituting  $\sqrt{x^2 + 7x + 9} = u$  we get

$$u^2 + u - 12 = 0$$

$$\text{Or } (u + 4)(u - 3) = 0$$

$$\text{So } u = -4 \text{ and } 3.$$

$$\text{So } \sqrt{x^2 + 7x + 9} = 3$$

$$\text{Squaring both sides we get } x^2 + 7x + 9 = 9 \text{ Or, } x^2 + 7x = 0 \text{ Or, } x(x + 7) = 0$$

$$\text{So } x = 0 \text{ and } -7$$

$$\text{Also } \sqrt{x^2 + 7x + 9} = -4 \text{ Or, } x^2 + 7x + 9 = 16 \text{ Or, } x^2 + 7x - 7 = 0$$

$$\text{So } x = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 1 \cdot (-7)}}{2 \cdot 1} \text{ Or, } x = \frac{-7 \pm \sqrt{49 + 28}}{2} \text{ So } x = \frac{-7 \pm \sqrt{77}}{2}$$

**Illustration 159.**

$$\text{Solve: } (x^2 + 3x)^2 + 2(x^2 + 3x) = 24$$

**Solution:**

$$\text{Let } x^2 + 3x = A,$$

$$(x^2 + 3x)^2 + 2(x^2 + 3x) = 24 \text{ Or, } A^2 + 2A - 24 = 0$$

$$\text{Or, } A^2 + 6A - 4A - 24 = 0$$

$$\text{Or, } A(A + 6) - 4(A + 6) = 0$$

$$\text{Or, } (A + 6)(A - 4) = 0$$

$$\text{Either, } A = x^2 + 3x = -6 \text{ Or } A = x^2 + 3x = 4$$

Either,  $x^2 + 3x + 6 = 0$  Or,  $x^2 + 3x - 4 = 0$

From  $x^2 + 3x + 6 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(3) \pm \sqrt{(3)^2 - 4.1.6}}{2.1} = \frac{-3 \pm \sqrt{9 - 24}}{2} = \text{Not feasible because } \sqrt{9 - 24} = \sqrt{-15} \text{ which is}$$

Imaginary Number.

From  $x^2 + 3x - 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(3) \pm \sqrt{(3)^2 - 4.1.(-4)}}{2.1} = \frac{-3 \pm \sqrt{9 + 16}}{2} = -4 \text{ and } 1$$

### Illustration 160.

Find 'k' for which roots are equal of  $4x^2 - 12x + k = 0$

**Solution:**

$$D = b^2 - 4ac = 0$$

$$\text{Or, } (-12)^2 - 4 \times 4k = 0$$

$$\text{Or, } 144 = 16k$$

$$\text{Or, } k = 9$$

### Illustration 161.

If difference between the roots of the equation  $x^2 - kx + 8 = 0$  is 0, find 'k'

**Solution:**

$\alpha$  &  $\beta$  be two roots.

$$\alpha + \beta = \frac{-(-k)}{1} = k \text{ \& } \alpha\beta = 8$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = k^2 - 4 \times 8 = 0$$

$$\text{Or, } k^2 = 32 \quad \text{Or } k = \pm(4\sqrt{2})$$

### Illustration 162.

If one of the root of the equation  $x^2 + px + a$  is  $\sqrt{3}+2$ , then find value of p & a.

**Solution:**

Roots are  $\sqrt{3}+2, 2-\sqrt{3}$ .

$$\text{Equation is } x^2 - (\text{Sum of roots})x + \text{product of roots} = 0$$

$$\text{Or, } x^2 - 4x + (4 - 3) = 0$$

$$\text{Or, } x^2 - 4x + 1 = 0, \text{ hence } p = -4, a = 1$$

### Illustration 163.

If arithmetic mean between roots of quadratic equation is 8 and the geometric mean between them is 5. Find equation.

**Solution:**

$\alpha$  &  $\beta$  be two roots.

$$\alpha + \beta = 2 \times 8 = 16, \quad \alpha\beta = 25.$$

$$\text{Equation is } x^2 - 16x + 25 = 0.$$

### Illustration 164.

If the ratio of the equation.  $4x^2 - 6x + p = 0$  is 1:2, find value of 'p'.

**Solution:**

let  $\alpha$  is common root

$$\alpha + 2\alpha = \frac{-(-6)}{4} = \frac{6}{4} \quad \text{or} \quad \alpha = \frac{1}{2}$$

$$\alpha \cdot \frac{1}{2\alpha} = \frac{p}{4} \quad \text{or} \quad p = 2.$$

**Illustration 165.**

If one root of equation  $x^2 - 3x + k = 0$  is 2, find 'k'.

**Solution:**

2 is root of the equation

$$2^2 - 3 \times 2 + k = 0 \quad \text{or} \quad k = 2.$$

**Illustration 166.**

Roots of equation  $3x^2 - 14x + k = 0$  will be reciprocal of each other, find 'k'.

**Solution:**

let one root =  $\alpha$ , other root =  $\frac{1}{\alpha}$

Product of roots =  $\frac{c}{a}$

$$\alpha \cdot \frac{1}{\alpha} = \frac{k}{3} \quad \text{or} \quad k = 3.$$

**Illustration 167.**

Find positive value of k for which the equations  $x^2 + kx + 64 = 0$  and  $x^2 - 8x + k = 0$  will have real roots.

**Solution:**

for equation (1)

$$b^2 - 4ac \geq 0$$

$$k^2 \geq 4 \times 1 \times 64 \quad \text{or,} \quad k \geq 16$$

for equation (2)

$$(-8)^2 \geq 4 \times 1 \times k \quad \text{or} \quad k \leq 16$$

$$\text{From (1) \& (2)} \quad k = 16.$$

**Exercise:****Theoretical Questions**⊙ **Multiple Choice Questions (MCQ)**

In a customer feedback survey of 60, it was found Coke was liked by 24, Fanta by 27, Coca Cola by 11, Thums Up by 9. While, 18 liked both Fanta and Coke and 15 liked both Thums Up and Coca Cola and 11 liked all.

Based on the above information answer Q. Nos. 1 to 10.

1. How many people liked any one of the products?
  - (a) 33
  - (b) 18
  - (c) 11
  - (d) 27
2. How many people liked Fanta?
  - (a) 18
  - (b) 11
  - (c) 24
  - (d) 27
3. How many people did not like any of the products?
  - (a) 33
  - (b) 13
  - (c) 9
  - (d) 15
4. How many people liked either Fanta or Thums Up?
  - (a) 27
  - (b) 36
  - (c) 37
  - (d) 28
5. How many people liked Fanta, Coke or Thums Up?
  - (a) 42
  - (b) 53
  - (c) 41
  - (d) 54

6. How many people liked Mazaa?
  - (a) 0
  - (b) Not available as product (None of them)
  - (c) Zero
  - (d) All of the above
7. How many people liked any two products?
  - (a) 18
  - (b) 15
  - (c) 33
  - (d) 22
8. How many people liked Thums Up, Coke, Coca Cola and Fanta?
  - (a) 11
  - (b) 49
  - (c) 60
  - (d) 50
9. How many people liked all the products?
  - (a) 60
  - (b) 11
  - (c) 09
  - (d) 12
10. How many people liked either Coke or Thums Up?
  - (a) 24
  - (b) 11
  - (c) 09
  - (d) 33
11. If Set A = {Q, W, E, R, T, Y} and Set B = {B, G, R, E, O, K}, find (A-B).
  - (a) Set (A-B) = {Q, W, T, Y}
  - (b) Set (A-B) = { B, G, O, K}
  - (c) Set (A-B) = { E, R}
  - (d) None of the Above
12. If A={W, N, M, I}; B={Q, E, H, J}; C= {I, L, E, R}. Find  $A \cup (B \cap C)$ .
  - (a) {W, Q, E, H, J}
  - (b) {Q, I, L, E, R}

- (c)  $\{E, W, N, M, I\}$   
 (d)  $\{T, I, L, E, R\}$
13. If Set  $B = \{B, G, R, E, O, K\}$  is a subset of Set  $A = \{Q, E, R, T, Y, U, I, O, P, K, G, B\}$ , find  $B'$ .
- (a) Set  $B' = \{B, G, R, E\}$   
 (b) Set  $B' = \{Q, E, G, B\}$   
 (c) Set  $B' = \{E, R, T, Y\}$   
 (d) Set  $B' = \{Q, T, Y, U, I, P\}$
14. If Set  $A = \{8, 9, 7, 5, 6, 2\}$  and Set  $B = \{1, 4, 9, 3, 8, 2\}$ , find the union of B and A.
- (a) Set  $(B \cup A) = \{8, 9, 7, 5, 6, 2, 1, 4, 9, 3, 8, 2\}$   
 (b) Set  $(B \cup A) = \{8, 9, 2\}$   
 (c) Set  $(B \cup A) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$   
 (d) None of the Above
15. If  $P_1 = \{78, 77, 76, 75\}$ ;  $P_2 = \{71, 72, 73, 74\}$ ;  $P_3 = \{80, 79, 70, 69\}$ .  
 Find  $(P_1 \cap P_2) \cup (P_2 \cap P_3)$ .
- (a)  $(P_1 \cap P_2) \cup (P_2 \cap P_3) = \{ \}$   
 (b)  $(P_1 \cap P_2) \cup (P_2 \cap P_3) = \{ \}$   
 (c) Both 'a' and 'b'  
 (d) None of the Above
16. If  $A = \{W, R, Y, I\}$ ;  $B = \{Q, E, T, U\}$ ;  $C = \{Q, W, E, R\}$ . Find  $A \cap (B \cup C)$ .
- (a)  $A \cap (B \cup C) = \{W, R\}$   
 (b)  $A \cap (B \cup C) = \{Q, E\}$   
 (c)  $A \cap (B \cup C) = \{Y, I\}$   
 (d)  $A \cap (B \cup C) = \{T, U\}$
17. If Set  $A = \{Q, W, E, R, T, Y\}$  and Set  $B = \{B, G, R, E, O, K\}$ , find  $(B-A)$ .
- (a) Set  $(B-A) = \{Q, W, T, Y\}$   
 (b) Set  $(B-A) = \{B, G, O, K\}$   
 (c) Set  $(B-A) = \{E, R\}$   
 (d) None of the Above
18. If  $A = \{8, 11, 14, 17\}$ ;  $B = \{5, 7, 9, 15\}$ ;  $C = \{19, 21, 17, 15\}$ . Find  $A \cup (B \cap C)$ .
- (a)  $A \cup (B \cap C) = \{8, 11, 14, 15, 17\}$   
 (b)  $A \cup (B \cap C) = \{8, 5, 7, 9, 15\}$   
 (c)  $A \cup (B \cap C) = \{11, 19, 21, 17, 15\}$   
 (d)  $A \cup (B \cap C) = \{11, 19, 21, 15, 8\}$

19. In Venn diagram, Universal Set is represented by \_\_\_\_\_.
- Stars
  - Squares
  - Rectangle
  - Circles
20. According to De Morgan's Law:  $(B \cap C)' =$  \_\_\_\_\_.
- $B' \cup C'$
  - $B' \cap C'$
  - $B \cup C'$
  - $B' \cap C$
21. Find the value of:  $3^9 \times 3^3$
- $3^{12}$
  - 531441
  - $19683 \times 27$
  - All of the above
22. Find the value of  $(2.85)^t$ , when  $0.0016 = (1/5)^{4t}$ .
- 2.85
  - $(2.85)^2$
  - $(2.85)^{-1}$
  - Either 'b' or 'c'
23. When the value of  $p > m$ , simplify  $q^p \times q^m$ .
- $q^{m+p}$
  - $q^{m-p}$
  - $q^{m \times p}$
  - None of the above
24. What will be the value of  $(9^3)^2$  ?
- $59004 \times 9$
  - $59049 \times 9$
  - $49005 \times 9$
  - $49059 \times 9$
25. Find the value of  $11^{78}/11^{81}$  ?
- $11^3$
  - $1 / 11^3$

- (c) 1331  
(d) -1331
26. What will be the value of  $3^6 \times 3^4 \times 3^{-2} \times 3^{-3} \times 3^6$  ?  
(a)  $3^{21}$   
(b)  $3^{16}$   
(c)  $3^{19}$   
(d)  $3^{11}$
27. Find the value of  $6^3 \times 6^{-2} \times 6^{-5} \times 6^4$   
(a) 0  
(b) 1  
(c) 6  
(d) 216
28. Find the value of  $b$  when  $\left(\frac{a}{b}\right)^{a/b} = (a)^{a/b-1}$  and  $a = 2b$   
(a) 2  
(b)  $2a$   
(c)  $2^2$   
(d)  $a^2$
29. What will be the value of  $(7^3)^2$ ?  
(a) 117649  
(b) 196471  
(c) 16807  
(d) 18607
30. What will be the value of  $9^6 \times 9^{-4} \times 9^2 \times 9^3 \times 9^6$  ?  
(a)  $9^{13}$   
(b)  $9^{21}$   
(c) Both 'a' and 'b'  
(d) None of the above
31. Find the value of  $(9^2/3^3) \times (3^4/9)$ ?  
(a)  $3^4$   
(b)  $3^5$   
(c)  $3^3$   
(d)  $3^6$

32. Find the value of  $(1.25)^q$ , when  $(1/3)^{4q} = (6561)^{-1}$
- (a)  $1.5 + 0.0625$
  - (b)  $1.5625$
  - (c)  $(1.25)^2$
  - (d) All of the above
33. Simplify  $3(a^1)^2 b^1 / 6b^2(a^3)^1$
- (a)  $1/(2ab)$
  - (b)  $\frac{1}{2} \times ab$
  - (c)  $2ab$
  - (d) None of the above
34. Evaluate  $\frac{2^{1/3} \cdot 8^{2/3} \cdot 6^{-5/4} \cdot 3^{-3/4} \cdot 9}{\sqrt[3]{16}}$
- (a)  $\frac{1}{\sqrt[4]{2}}$
  - (b) 2
  - (c)  $\sqrt[4]{2}$
  - (d) None of the above
35. Simplify  $7^{7/9} \times 7^{2/9} \times 7^{4/9} \times 7^{-1} \times 7^{-4/9}$
- (a) 7
  - (b) 1
  - (c) 0
  - (d) None of the above
36. When  $a^x = b^y = c^z$  and  $b^2 = ac$  then  $1/x, 1/y, 1/z$  should be in ———
- (a) G.P
  - (b) A.P
  - (c) H.P
  - (d) None of the above
37. What will be the value of  $7^{-5} \times 7^{-8} \times 7^4 \times 7^3 \times 7^6$  ?
- (a) -7
  - (b) 1
  - (c) 7
  - (d) None of the above

38. Find the value of  $(3^3/9^2) \times (9/3^4)$ ?
- (a)  $3^{-1}$
  - (b)  $3^{-2}$
  - (c)  $3^{-3}$
  - (d)  $3^{-7}$
39. What will be the value of  $(6^3 \times 6^2)/(6^{-3} \times 6^4)$  ?
- (a)  $(6^2)^2$
  - (b)  $6^5$
  - (c) 6
  - (d) None of the above
40. Find the value of  $19^9/19^2$
- (a)  $19^2$
  - (b)  $19^7$
  - (c)  $19^9$
  - (d)  $19^0$
41. Express  $\log_9 1 = 0$  in exponential form.
- (a)  $9^1 = 0$
  - (b)  $9^0 = 1$
  - (c)  $1^9 = 1$
  - (d)  $0^9 = 0$
42. Express  $4^3 = 64$  in logarithmic form.
- (a)  $\log_4 6 = 3$
  - (b)  $\log_{64} 3 = 4$
  - (c)  $\log_4 64 = 3$
  - (d)  $\log_3 64 = 4$
43. Solve :  $\log_{10}(8y + 3) = 3$
- (a)  $\frac{1003}{8}$
  - (b)  $\frac{997}{8}$

(c)  $\frac{1008}{3}$

(d)  $\frac{992}{3}$

44. If  $\log_4(t-1)+1 = 2 \times \log_4 t$ , find the value of  $t$ .

(a) 1

(b) 0

(c) 4

(d) 2

45. Find  $t$  when  $\log_3 t = 6$

(a) 729

(b) 27

(c) 81

(d) 243

46. Find the logarithm of 324 with the base  $3\sqrt{2}$ .

(a) -4

(b) 0

(c) -2

(d) 4

47. Find the value of  $z$  :  $100z = z^{\log_{10} z}$

(a) 100

(b)  $\frac{1}{10}$

(c) Both 'a' and 'b'

(d) None of the above

48. For the logarithms to be in A. P. Series, the positive numbers should be following \_\_\_\_.

(a) H. P.

(b) G. P.

(c) A. P.

(d) None of the above.

49. If  $\log_y \frac{1}{2} = \frac{1}{3}$ , what will be the value of base?
- (a)  $\frac{1}{2}$
  - (b)  $\frac{1}{4}$
  - (c)  $\frac{1}{8}$
  - (d)  $-\frac{1}{2}$
50. Find  $t$  when  $\log_t 3125 = 5$
- (a) 5
  - (b) 125
  - (c) 25
  - (d) 625
51. State True or False:  $\log 2 + 2 \log \frac{25}{24} = 3 \log \frac{81}{80} + 7 \log \frac{10}{9}$
- (a) True
  - (b) False
  - (c) Not possible
  - (d) None of the above
52. State True or False:  $\log_2 10 + \log_8 125 = 1$
- (a) Not possible
  - (b) True
  - (c) May be true
  - (d) None of the above
53. State True or False:  $\log_3 32 = 2.15$  when  $\log 2 = 0.30103$ .
- (a) False
  - (b) True
  - (c) LHS > RHS
  - (d) LHS < RHS

54. Find the value of the logarithm of  $2^{\text{nd}}$  number (b) for 3 consecutive numbers (a, b, c).
- (a)  $\log(1 + ac)$
  - (b)  $\frac{1}{2} \times 2 \times \log(1 + ac)$
  - (c)  $\frac{1}{2} \times \log(1 + ac)$
  - (d)  $2 \times \log(1 + ac)$
55. If  $\log_y 32 = 10$ , then which of the following is the value of  $y$  ?
- (a) 4
  - (b) 2
  - (c)  $\sqrt{4}$
  - (d)  $\sqrt{2}$
56. Find the base when 3 is the logarithm of 729.
- (a) 7
  - (b) 11
  - (c) 9
  - (d) 3
57. If  $\log(7y - 5) = 2$ , find the value of  $y$ .
- (a) 15
  - (b) 10
  - (c) 08
  - (d) 07
58. For any series of positive numbers which are in G. P., their logarithms would be in \_\_\_\_\_.
- (a) G. P.
  - (b) A. P.
  - (c) H. P.
  - (d) None of the Above
59. Solve:  $\log(9r - 2) = 2$
- (a)  $\frac{102}{9}$
  - (b)  $\frac{98}{9}$
  - (c)  $\frac{109}{2}$
  - (d)  $\frac{91}{2}$

60. Find the logarithm of 125 to the base  $5\sqrt{5}$
- (a) 6
  - (b) 4
  - (c) 3
  - (d) 2
61. Compute  $\frac{12!}{(8! \times 4!)}$
- (a) 3960
  - b. 495
  - c. 99
  - d. 440
62. Find the value of  $v$ , if  $(v-1)! \times 20 = (v+1)!$
- (a) 0
  - (b) 2
  - (c) 4
  - ( ) 5
63. Determine the number of ways in which a digit and a vowel can be selected from 10 digits and 26 letters of Alphabet?
- (a) 50
  - (b) 210
  - (c) 105
  - (d) None of the Above
64. Find the LCM of  $\{12!, 14!, 13!\}$
- (a) 11!
  - (b) 15!
  - (c) 12!
  - (d) 14!
65. When 9 local trains are running between Haldia and Burdwan. In how many ways can a passenger travel from Burdwan to Haldia and return by a different train?
- (a) 72 ways.
  - (b) 70 ways.
  - (c) 64 ways.
  - (d) 60 ways.

66. If  ${}^n P_6 : {}^n P_5 = 1:3$ , find the value of  $n$ .
- (a) 3
  - (b) 4
  - (c) 7
  - (d)  $\frac{16}{3}$
67. Find the value of  $\frac{2}{36!} - \frac{1}{35!} - \frac{1}{35!}$
- (a)  $-35 / 18!$
  - (b)  $-1 / 18!$
  - (c)  $-2 / 36!$
  - (d)  $-70 / 36!$
68. Find the number of permutations for 15 scooters if 3 scooters are to be considered at a time.
- (a) 2730
  - (b) 2370
  - (c) 2184
  - (d) 2814
69. Find the value of  $\frac{10! - 9!}{8!}$
- (a) 81
  - (b) 80
  - (c) 10
  - (d) 9
70. How many ways can 5 drivers refill their tanks from 5 refills, assuming no refills in the fuel station remain unused?
- (a) 24 ways.
  - (b) 120 ways.
  - (c) 5 ways.
  - (d) 60 ways.
71. In how many ways 8 cardholders stand in a queue at fair price shop?
- (a) 40320 ways
  - (b) 40230 ways
  - (c) 20430 ways
  - (d) 20340 ways

72. Find the number of permutations for 11 bikes if 5 bikes are to be taken at a time.
- (a) 54540
  - (b) 55440
  - (c) 44550
  - (d) 45450
73. Find the value of  $\frac{9}{7!} - \frac{3}{6!} + \frac{12}{7!}$
- (a) 0
  - (b)  $1/7!$
  - (c)  $-1/7!$
  - (d) None of the above
74. If  ${}^n P_7 : {}^n P_8 = 1:4$ , find the value of  $n$ .
- (a) 12
  - (b) 10
  - (c) 11
  - (d) 13
75. Find the value of  $a$ , if  $(a-2)! \times 24 = (a+1)!$
- (a) 0
  - (b) 2
  - (c) 4
  - (d) 3
76. What would be the factorial notation for:  $11 \times 10 \times 9 \times 8 \times 7$
- (a)  $11! / 6!$
  - (b)  $11! / 5!$
  - (c)  $10! / 6!$
  - (d)  $10! / 5!$
77. How many ways can 8 people get vaccinated from 8 vaccinators, assuming no vaccinator is idle?
- (a) 40320 ways.
  - (b) 5040 ways.
  - (c) 5760 ways.
  - (d) 35280 ways.

78. In how many different ways can 4 different cars, one of each of the 4 manufacturers, be parked in a parking lane?
- (a) 20 ways  
 (b) 22 ways  
 (c) 24 ways  
 (d) 26 ways
79. Find the value of  $\frac{3}{13!} - \frac{2}{12!} - \frac{2}{12!}$
- (a)  $-49 / 13!$   
 (b)  $-30 / 13!$   
 (c)  $-25 / 12!$   
 (d)  $-50 / 12!$
80. In how many ways 6 customers stand in a queue for depositing cash in bank?
- (a) 680 ways  
 (b) 480 ways  
 (c) 600 ways  
 (d) 720 ways
81. Examine the nature of the roots for the following equation  $16x^2 - 24x + 9 = 0$
- (a) Real and unequal  
 (b) Unreal and Imaginary  
 (c) Real and Rational  
 (d) None of the above
82. For what value of  $c$ , would the product of roots be zero?
- (a)  $a = 1, b = 1, c = 1$   
 (b)  $a = 1, b = 0, c = 1$   
 (c)  $a = 1, b = 1, c = 0$   
 (d)  $a = 1, b = 2, c = 2$
83. Form the equation whose roots are 9, -4
- (a)  $x^2 + 5x - 36 = 0$   
 (b)  $x^2 - 5x - 36 = 0$   
 (c)  $x^2 - 5x + 36 = 0$   
 (d)  $x^2 + 5x + 36 = 0$

84. Form quadratic equation with roots as  $a - t, a + t$

(a)  $x^2 - 2ax + a^2 - t^2 = 0$

(b)  $x^2 + 2ax + a^2 - t^2 = 0$

(c)  $x^2 - 2ax - a^2 + t^2 = 0$

(d)  $x^2 + 2ax - a^2 + t^2 = 0$

Solve Q85 – Q90 using the following information

When  $\alpha$  and  $\beta$  are the roots of  $5x^2 - 7x + 9 = 0$  then find the values of:

85.  $1/\beta + 1/\alpha$

(a)  $7/9$

(b)  $-7/9$

(c)  $9/7$

(d)  $-9/7$

86.  $\beta^2 + \alpha^2$

(a)  $41/25$

(b)  $-41/25$

(c)  $139/25$

(d)  $-139/25$

87. Sum of the Roots

(a)  $9/5$

(b)  $5/9$

(c)  $-7/5$

(d)  $7/5$

88. Product of the Roots

(a)  $-7/5$

(b)  $7/5$

(c)  $-9/5$

(d)  $9/5$

89.  $\alpha/\beta + \beta/\alpha$

(a)  $369/5$

(b)  $-41/45$

- (c)  $-369/5$   
 (d)  $41/45$
90.  $\alpha + \beta + 4$   
 (a)  $27/5$   
 (b)  $29/5$   
 (c)  $28/5$   
 (d)  $24/5$
91. If  $b^2 - 4ac > 0$ , is a perfect square, the nature of roots would be  
 (a) Real and Equal  
 (b) Imaginary  
 (c) Unreal  
 (d) Real and Unequal
92. For what values of  $a$  and  $b$ , the product of roots would be equal to  $c$ ?  
 (a)  $a = c \neq b$   
 (b)  $a \neq b = c$   
 (c)  $a = 1$   
 (d)  $a \leq 0$
93. Choose the correct form (From the alternatives given below) in which quadratic equation is generally written -  
 (a)  $ax^2 + bx + c = 0$   
 (b)  $x^2 + ax - b = 0$   
 (c)  $ax^2 - bx + c = 0$   
 (d)  $x^2 - bx + c = 0$
94. Choose the correct condition for any equation to be Quadratic equation-  
 (a) 5 constants and all constants  $> 0$ .  
 (b) 5 constants and all constants  $< 0$ .  
 (c) 3 constants and  $a > 0$ .  
 (d) 3 constants and  $a \neq 0$ .
95. Which one of the following is Discriminant of a quadratic equation?  
 (a)  $-b + b^2 - 4ac$   
 (b)  $-b - b^2 - 4ac$   
 (c)  $b^2 - 4ac$   
 (d)  $\sqrt{b^2 + 4ac}$

96. When are nature of roots real rather than imaginary?
- (a) If Discriminant is negative
  - (b) If Discriminant is less than zero
  - (c) If Discriminant is more than or equal to zero
  - (d) None of the above
97. For what values of  $a$  and  $c$ , value of sum of the roots would be equal to  $b$ .
- (a)  $a = 1, c = n$
  - (b)  $a = -1, c = n$
  - (c)  $a = n, c = -1$
  - (d)  $a = n, c = 1$
98. For any sum of roots of quadratic equation, ' $b$ ' represents -
- (a) Coefficient of  $x$
  - (b) Constant term
  - (c) Coefficient of  $x^2$
  - (d) All of the above
99. Find the square of the difference of the roots of  $115 + 5(x^2 - 12x) = 0$
- (a) 24
  - (b) 48
  - (c) 26
  - (d) 52
100. For any sum of roots of quadratic equation, ' $a$ ' represents -
- (a) Coefficient of  $x$
  - (b) Coefficient of  $x^2$
  - (c) Constant term
  - (d) None of the above

⊙ **State True or False**

1. The statement  $(A \cap B)^1 = A^1 \cup B^1$  is true (or) False
2. If the set A has 4 elements, B has 3 elements then the number of elements in  $A \times B$  is 12 elements
3. The Statement  $\{2\} \not\subset \{2, 3, 5\}$  is true (or) False
4. The statement  $\{1\} \subset \{1, 2, 3\}$  is true (or) False
5. The statement “Equivalent sets” are always equal is True (or) False
6. If  $x = \sqrt[3]{\sqrt{2+1}} - \sqrt[3]{\sqrt{2-1}}$  then the value of  $x^3 + 3x$  is 2
7. If  $x = 5 + 2\sqrt{6}$  and  $xy = 1$ , then  $\frac{1}{x^2} + \frac{1}{y^2}$  is 89
8. If  $2^{x+2y} = 2^{2x-y} = \sqrt{8}$ , then  $x = \frac{9}{10}$  and  $y = \frac{10}{3}$
9. If  $x = 2 + \sqrt{5}$ , then  $x^3 + 3x^2 - 29x$  is 7
10.  $\log 3 + \log 5$  is  $\log 15$
11. The value of  $\log_2 \log_2 \log_3 81 = 2$
12. The logarithm of 324 to base  $\frac{1}{3\sqrt{2}}$  is  $-4$
13. The logarithms with base 10 are called Natural logarithm
14. The logarithms with base e are called common Logarithm
15. The logarithm of one to any base is zero
16. The logarithm of a number which is not equal to one with itself as base is zero
17. The integral part of the value of logarithm of a number is called characteristic
18. The decimal part of the value of logarithm of a number is called mantissa
19. The logarithm of same number for different bases are different
20. A polygon has 44 diagonals then the number of its sides are 8
21. The total number of 9 digit numbers which have all different digits is  $9 \times 9$
22. There are 8 questions in an examination paper and each question has an alternative. The number of ways in which a student can give his answer is 6561
23. In a football competition there were 153 matches. A match occurs between two teams. The total numbers of teams took part in the competition is 18
24. If  $n_{C_n} = 1$ , then  $0! = 1$
25. If the roots of the equation  $2x^2 + 8x + c = 0$  are equal then  $c = 8$
26. The g. c. d of the equations  $2x^2 - x - 1 = 0$  and  $4x^2 + 8x + 3 = 0$  is  $3x + 1$
27.  $x^2 - 4x - 1 = 0$  is the quadratic equation whose roots are 2 and 5

28. The roots of the equation  $(x-4)^2(x-2)(x+4) = 0$  are 4, -4, 2 and -2

29. The degree of the expression  $3x^5 + xyz^2 + y^3$  is 3

⊙ **Fill in the Blanks**

1. If  $A = \{1, 2, 3, 4\}$ ,  $B = \{2, 4, 6\}$ , then  $A \Delta B$  is \_\_\_\_\_ [  $A \Delta B = (A-B) \cup (B-A)$  and is known as the symmetric difference of two sets ]

2. If A and B are two sets then  $A \cap (B-A)$  is \_\_\_\_\_

3. If  $A = \{1, 2, 3\}$ ,  $B = \{2, 3, 4\}$ ,  $C = \{1, 2, 5, 6\}$  then  $A \cup (B \cap C)$  is \_\_\_\_\_

4. If A and B are two sets then  $A \cap B = A \cup B$  if and only if \_\_\_\_\_

5. If A and B are two disjoint sets then  $n(A \cup B)$  is equal to \_\_\_\_\_

6. If A has 2 elements, B has 4 elements and  $A \subset B$  then  $A \cap B$  has \_\_\_\_\_ elements

7. If A and B are the two sets of positive and negative integers respectively then  $A \cup B$  is \_\_\_\_\_

8. If  $a^{1/3} + b^{1/3} + c^{1/3} = 0$ , then  $(a+b+c)^3 =$  \_\_\_\_\_

9. If  $y = x^{1/3} - x^{-1/3}$ , then  $y^3 + 3y =$  \_\_\_\_\_

10. If  $a = 2 + \sqrt[3]{2} + \sqrt[3]{4}$ , then  $a^3 - 6a^2 + 12a =$  \_\_\_\_\_

11. If  $64^x = 2\sqrt{2}$ , then  $x =$  \_\_\_\_\_

12. If  $x = 8, y = 27$ , then the value of  $(x^{4/3} + y^{2/3})^{1/2} =$  \_\_\_\_\_

13. If  $\log_x 0.1 = \frac{1}{3}$  then the value of x is: \_\_\_\_\_

14. If  $\log_{32} x = 0.8$  then x is equal to: \_\_\_\_\_

15. If  $\log_4 x + \log_2 x = 6$  then x is equal to: \_\_\_\_\_

16. If  $\log_8 x + \log_8 \left(\frac{1}{6}\right) = \frac{1}{3}$  then the value of x is: \_\_\_\_\_

17. If  $\log 2 = 0.30103$ , then the number of digits in  $4^{50}$  is: \_\_\_\_\_

18. If  $\log 2 = 0.30103$ , then the number of digits in  $5^{20}$  is: \_\_\_\_\_

19. The value of  $\log_{(-\frac{1}{3})} 81$  is equal to: \_\_\_\_\_

20. The value of  $\log_{2\sqrt{3}}(1728)$  is equal to: \_\_\_\_\_

21. The value of  $\log_2(\log_5 625)$  is: \_\_\_\_\_

22. The value of  $\frac{1}{3} \log_{10} 125 - 2 \log_{10} 4 + \log_{10} 32$  is: \_\_\_\_\_

23. There are 10 lamps in a room. Each one of them can be switched on independently. The number of ways in which the hall can be illuminated is \_\_\_\_\_

24. These are 11 questions in an examination paper of mathematics. A candidate has to answer 6 questions of which the question under 1 is compulsory. The total number of selections of his answering in questions is \_\_\_\_\_
25. There are two groups in a question paper, each group contains 7 questions. A candidate has to answer questions but taking not more than 5 from any group. The total number of selections of 9 questions is \_\_\_\_\_
26. There are 10 points in plane and among them 4 are collinear. The total number of triangles formed by joining them is \_\_\_\_\_
27. Out of 18 points in plane, no three are in the same straight line except 5 points which are collinear. Then the number of straight lines obtained by joining them is \_\_\_\_\_
28. If the equations  $x^2 - 7x + 12 = 0$  and  $x^2 + mx + 5 = 0$  have common roots the value of  $m$  is equal to \_\_\_\_\_
29. The least positive value of  $m$  for which the equation  $x^2 + mx + 4 = 0$  has real roots \_\_\_\_\_
30. The value of  $m$  for which the difference between the roots of the equation  $x^2 + mx + 4 = 0$  is 2 are \_\_\_\_\_
31. If  $p, q$  are the roots of the equation  $f(x) = 6x^2 + x - 2 = 0$  the value of  $p/q$  is \_\_\_\_\_

**Answer:**

⊙ **Multiple Choice Questions (MCQ)**

Q. No.	Answer	Remarks
1	(d)	Like any of the product : $n(A \cup B \cup C \cup D) = n(A) + n(B) + n(C) + n(D) - n(A \cap B) - n(A \cap C) - n(A \cap D) - n(B \cap C) - n(B \cap D) - n(C \cap D) + n(A \cap B \cap C) + n(A \cap B \cap D) + n(A \cap C \cap D) + n(B \cap C \cap D) - n(A \cap B \cap C \cap D) = 24 + 27 + 11 + 9 - 18 - 15 - 11 = 27$
2	(d)	Like Fanta = $B = n(B) = 27$
3	(a)	Did Not Like any of the products : $n(A \cup B \cup C \cup D)' = n(S) - [(n(A) + n(B) + n(C) + n(D) - n(A \cap B) - n(C \cap D) + n(A \cap B \cap C \cap D))] = 60 - (24 + 27 + 11 + 9 - 18 - 15 - 11) = 60 - 27 = 33$
4	(b)	Liked either Fanta or Thums Up = $n(B \cup D) = n(B) + n(D) - n(B \cap D) = 27 + 9 - 0 = 36$
5	(a)	Liked Fanta, Coke or Thums Up = $n(A \cup B \cup D) =$ $n(A) + n(B) + n(D) - n(B \cap D) - n(A \cap B) - n(A \cap D) + n(A \cap B \cap D)$ $= 24 + 27 + 9 - 0 - 18 - 0 + 0 = 42$
6	(d)	Liked Mazaa: Since Mazaa is not available as a product, None of them liked Mazaa or it can be considered as Zero who liked Mazaa.
7	(c)	Liked any two products : $n(A \cap B) + n(C \cap D) = 18 + 15 = 33$ (Intersection of B&C, C&A, A&D, B&D & C & D are not taken into account because no information regarding no. of elements of these sets is provided)
8	(a)	Liked Thums Up, Coke, Coca Cola and Fanta : Comprises of all the products = Liked all the products : 11 people liked all.
9	(b)	Comprises of all the products = Liked all the products : 11 people liked all.
10	(d)	Either Coke or Thums Up : $n(A \cup D) = n(A) + n(D) - n(A \cap D) = 24 + 9 - 0 = 33$
11	(a)	Set A = {Q, W, E, R, T, Y} and Set B = {B, G, R, E, O, K} Set (A-B) = {Q, W, T, Y} Since, Q, W, T, Y are present in Set A but not present in Set B.
12	(c)	A = {W, N, M, I} ; B = {Q, E, H, J} ; C = {I, L, E, R} $B \cap C = \{E\}$ $A \cup (B \cap C) = \{W, N, M, I, E\}$

13	(d)	Set B = {B, G, R, E, O, K} is a subset of Set A = {Q, E, R, T, Y, U, I, O, P, K, G, B} $B' = \text{Set}(A - B) = \{Q, T, Y, U, I, P\}$
14	(c)	Set A = {8, 9, 7, 5, 6, 2} and Set B = {1, 4, 9, 3, 8, 2} Union of B and A = {1, 2, 3, 4, 5, 6, 7, 8, 9}
15	(c)	$P_1 = \{78, 77, 76, 75\}$ ; $P_2 = \{71, 72, 73, 74\}$ ; $P_3 = \{80, 79, 70, 69\}$ . $(P_1 \cap P_2) \cap (P_2 \cap P_3) = P_2 \cap (P_1 \cap P_3)$ $P_1 \cup P_3 = \{69, 70, 75, 76, 77, 78, 79, 80\}$ $(P_1 \cap P_2) \cup (P_2 \cap P_3) = \{ \} = \{\phi\}$
16	(a)	$A = \{W, R, Y, I\}$ ; $B = \{Q, E, T, U\}$ ; $C = \{Q, W, E, R\}$ $B \cup C = \{Q, W, E, R, T, U\}$ $A \cap (B \cup C) = \{W, R\}$
17	(b)	Set A = {Q, W, E, R, T, Y} and Set B = {B, G, R, E, O, K} Set (B - A) = {B, G, O, K} Since, B, G, O, K are present in Set B but not present in Set A.
18	(a)	$A = \{8, 11, 14, 17\}$ ; $B = \{5, 7, 9, 15\}$ ; $C = \{19, 21, 17, 15\}$ $B \cap C = \{15\}$ $A \cup (B \cap C) = \{8, 11, 14, 15, 17\}$
19	(c)	In Venn Diagram, Universal Set is represented by <u>Rectangles</u> .
20	(a)	According to De Morgan's Law : $(B \cap C)' = B' \cup C'$
21	(d)	$3^9 \times 3^3 = 3^{9+3} = 3^{12} = 531441$ (19683×27) All of the above
22	(a)	$0.0016 = (1/5)^{4t}$ , It as $16/10000 = 1/625 = 1/(5^4) = (1/5)^4$ So, $(1/5)^4 = (1/5)^{4t}$ or, $4=4t$ or, $t=1$ $2.85^t = 2.85^1 = 2.85$
23	(a)	$q^p \times q^m = q^{m+p}$
24	(b)	$(9^3)^2 = 9^6 = 531441 = 59049 \times 9$
25	(b)	$11^{78}/11^{81} = 11^{78-81} = 1/11^3$
26	(d)	$3^6 \times 3^4 \times 3^{-2} \times 3^{-3} \times 3^6 = 3^{6+4-2-3+6} = 3^{11}$
27	(b)	$6^3 \times 6^{-2} \times 6^{-5} \times 6^4 = 6^{3+(-2)+(-5)+4} = 6^{-7} = 6^0 = 1$
28	(a)	$\left(\frac{a}{b}\right)^{\frac{a}{b}} = (a)^{\frac{a}{b}-1}$ or, $\left(\frac{2b}{b}\right)^{\frac{2b}{b}} = (2b)^{\frac{2b}{b}-1}$ [As $a = 2b$ ] or, $2^2 = (2b)^{2-1}$ or, $2^2 = 2b$ or, $b = 2$
29	(a)	$(7^3)^2 = 7^6 = 117649$

30	(a)	$9^6 \times 9^{-4} \times 9^2 \times 9^3 \times 9^6 = 9^{6-4+2+3+6} = 9^{13}$
31	(c)	$(9^2/3^3) \times (3^4/9) = 3^{2 \times 2} / 3^3 \times 3^4/3^2 = 3^{4-3} \times 3^{4-2}$ $= 3^1 \times 3^2 = 3^{1+2} = 3^3$
32	(d)	$(1/3)^{4q} = 1/6561$ or, $(1/3)^{4q} = 1/3^8 = (1/3)^8$ So, $4q=8$ or, $q=2$ $(1.25)^q = (1.25)^2 = 1.5625 = (1.50+0.0625)$
33	(a)	$3(a^1)^2 b^1 / 6b^2(a^3)^1 = (3/6) a^{1 \times 2} b^1 / a^{3 \times 1} b^2$ $= (1/2) a^2 b^1 / a^3 b^2 = (1/2) a^{2-3} b^{1-2} = (1/2) a^{-1} b^{-1} = 1/(2ab)$
34	(a)	$2^{\frac{1}{3}} \cdot 8^{\frac{2}{3}} \cdot 6^{\frac{-5}{4}} \cdot 3^{\frac{-3}{4}} \cdot 9 / \sqrt[3]{16}$ $= 2^{\frac{1}{3}} \cdot (2^3)^{\frac{2}{3}} \cdot (2 \cdot 3)^{\frac{-5}{4}} \cdot 3^{\frac{-3}{4}} \cdot (3^2) / (2^4)^{\frac{1}{3}} = 2^{\frac{1}{3}} \cdot 2^2 \cdot 2^{\frac{-5}{4}} \cdot 3^{\frac{-5}{4}} \cdot 2^{\frac{-3}{4}} \cdot 3^2 / 2^{\frac{4}{3}}$ $= 2^{\frac{1}{3}+2-\frac{5}{4}-\frac{3}{4}} \cdot 3^{\frac{-5}{4}-\frac{3}{4}+2}$ $= 2^{\frac{4+24-15-16}{12}} \cdot 3^{\frac{-5-3+8}{4}}$ $= 2^{\frac{1}{4}} \cdot 3^0 = \left(\frac{1}{2}\right)^{\frac{1}{4}} = \frac{1}{\sqrt[4]{2}}$
35	(b)	$7^{7/9} \times 7^{2/9} \times 7^{4/9} \times 7^{-1} \times 7^{-4/9} = 7^{7/9+2/9+4/9-1-4/9} = 7^0 = 1$
36	(b)	$= a^x = b^y = c^z = K$ (Let) $\therefore a = k^{\frac{1}{x}}, b = k^{\frac{1}{y}}, c = k^{\frac{1}{z}}$ $b^2 = ac$ or, $\left(k^{\frac{1}{y}}\right)^2 = k^{\frac{1}{x}}, k^{\frac{1}{x}}$ or, $k^{\frac{2}{y}} = k^{\frac{1}{x}+\frac{1}{z}}$ So $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}$ i.e. $\frac{1}{x}, \frac{1}{y}$ & $\frac{1}{z}$ are in A.P.
37	(b)	$7^{-5} \times 7^{-8} \times 7^4 \times 7^3 \times 7^6 = 7^{-5-8+4+3+6} = 7^0 = 1$
38	(c)	$(3^3/9^2) \times (9/3^4) = 3^3 / 3^{2 \times 2} \times 3^2/3^4 = 3^{3-4} \times 3^{2-4}$ $= 3^{-1} \times 3^{-2} = 3^{-1-2} = 3^{-3}$
39	(a)	$(6 \times 6^2)/(6^{-3} \times 6^4) = (6^{3+2})/(6^{-3+4})$ $= (6^5)/(6^1) = 6^{5-1} = 6^4 = (6^2)^2$
40	(b)	$19^9 / 19^2 = 19^{9-2} = 19^7$
41	(b)	$\log_9 1 = 0$ in Exponential Form : $9^0 = 1$
42	(c)	$4^3 = 64$ in Logarithmic Form : $\log_4 64 = 3$

43	(b)	$\log_{10}(8y + 3) = 3$ $(8y + 3) = 10^3$ or, $8y + 3 = 1000$ or, $8y = 1000 - 3$ or, $8y = 997$ or, $y = \frac{997}{8}$
44	(d)	$\log_4(t - 1) + 1 = 2 \times \log_4 t$ or, $\log_4(t - 1) + \log_4 4 = \log_4 t^2$ or, $(t - 1) \times 4 = t^2$ or, $t^2 - 4t + 4 = 0$ or, $(t - 2)^2 = 0$ or, $t = 2$
45	(a)	$\log_3 t = 6$ $t = 3^6$ $t = 729$
46	(d)	Logarithm of 324 with the base $3\sqrt{2}$ $\log_{3\sqrt{2}} 324 = x$ $(3\sqrt{2})^x = 324$ or, $(3\sqrt{2})^x = (3\sqrt{2})^4$ or, $x = 4$
47	(c)	$100z = z^{\log_{10} z}$ or, $\log_{10}(100z) = \log_{10}(z^{\log_{10} z})$ or, $\log_{10} 100 + \log_{10} z = \log_{10} z \times \log_{10} z$ or, $\log_{10} \times 10^2 + \log_{10} z = (\log_{10} z)^2$ or, $2 \log_{10} + \log_{10} z = (\log_{10} z)^2$ or, $2 + x = x^2$ (Let $\log_{10} z = x$ ) or, $x^2 - x - 2 = 0$ or, $(x - 2)(x + 1) = 0$ $\therefore x = 2 \text{ \& } -1$ $\log_{10} z = 2$ or, $z = 10^2 = 100$ & $\log_{10} z = -1$ or, $z = (10)^{-1}$ or, $z = \frac{1}{10}$
48	(b)	For the logarithms to be in A. P. Series, the positive numbers should be following G.P.

49	(c)	$\log_y \frac{1}{2} = \frac{1}{3}$ $\frac{1}{2} = y^{\frac{1}{3}} \text{ or, } \left(\frac{1}{2}\right)^3 = \left(y^{\frac{1}{3}}\right)^3$ $\text{or, } y = \frac{1}{8}$
50	(a)	$\log_t 3125 = 5$ $3125 = t^5$ $\text{or, } 5^5 = t^5$ $\text{or, } t = 5$
51	(a)	$\text{LHS} = \log 2 + 2 \log \frac{25}{24} = \log 2 + 2 \log \frac{5^2}{8 \times 3} = \log 2 + 2[\log 5^2 - \log (8 \times 3)]$ $= \log 2 + 2 [2 \log 5 - \log 8 - \log 3] = \log 2 + 2 [2 \log 5 - 3 \log 2 - \log 3]$ $= \log 2 + 4 \log 5 - 6 \log 2 - 2 \log 3 = 4 \log 5 - 2 \log 3 - 5 \log 2$ $\text{R.H.S} = 3 \log \frac{81}{80} + 7 \log \frac{10}{9}$ $= 3 (\log 81 - \log 80) + 7 (\log 10 - \log 9)$ $= 3 [\log 3^4 - \log (16 \times 5)] + 7 [\log (5 \times 2) - \log 3^2]$ $= 3 [4 \log 3 - \log 16 - \log 5] + 7 [\log 5 + \log 2 - 2 \log 3]$ $= 3 [4 \log 3 - 4 \log 2 - \log 5] + 7 \log 5 + 7 \log 2 - 14 \log 3$ $= 12 \log 3 - 12 \log 2 - 3 \log 5 + 7 \log 5 + 7 \log 2 - 14 \log 3$ $= 4 \log 5 - 2 \log 3 - 5 \log 2$ <p>So, L.H.S = R.H.S. Hence true</p>
52	(a)	<p>Not possible</p> $\log_2 10 + \log_8 125 \neq 1$
53	(b)	<p>True</p> $\log_5 32 = 2.15 \text{ when } \log 2 = 0.30103$ $\text{L.H.S} = \log_5^{32} = \frac{\log_{10}^{32}}{\log_{10}^5} = \frac{\log_{10}^{2^5}}{\log_{10}^{\left(\frac{10}{2}\right)}} = \frac{5 \log_{10}^2}{\log_{10}^{10} - \log_{10}^2}$ $= \frac{5 \times 0.30103}{1 - 0.30103} = 2.15 = \text{R.H.S.}, \text{ Hence true.}$

54	(c)	Let the consecutive numbers be $a = x - 1$ , $b = x$ and $c = x + 1$ $\frac{1}{2} \times \log(1 + ac)$ $= \frac{1}{2} \log[1 + (x - 1)(x + 1)] = \frac{1}{2} \log x^2 = \log x = \log b$
55	(d)	$\log_y 32 = 10$ or, $y^{10} = 32$ or, $y^{10} = 2^5$ or, $y = 2^{5/10}$ or, $y = \sqrt{2}$
56	(c)	base when 3 is the logarithm of 729 $\log_x 729 = 3$ $729 = x^3$ or, $9^3 = x^3$ $\therefore x = 9$
57	(a)	$\log(7y - 5) = 2$ or, $\log_{10}(7y - 5) = 2$ or, $(7y - 5) = 10^2$ or, $(7y - 5) = 100$ or, $7y = 100 + 5$ or, $7y = 105$ or, $y = \frac{105}{7}$ or, $y = 15$
58	(b)	Let us consider a series of 3 positive numbers $\frac{a}{r}$ , $a$ and $ar$ which are in G.P. Logarithm of these numbers will be $\log\left(\frac{a}{r}\right)$ , $\log a$ and $\log(ar)$ But $\log\left(\frac{a}{r}\right) = \log a - \log r$ and $\log(ar) = \log a + \log r$ Hence logarithm of the 3 positive numbers in G.P. can be rewritten as $(\log a - \log r)$ , $\log a$ , $(\log a + \log r)$ which are in A.P with common difference $\log r$
59	(a)	$\log(9t - 2) = 2$ or, $\log_{10}(9t - 2) = 2$ or, $(9t - 2) = 10^2$ or, $(9t - 2) = 100$ or, $9t = 100 + 2$ or, $9t = 102$ or, $t = \frac{102}{9}$

60	(d)	<p>logarithm of 125 to the base <math>5\sqrt{5}</math></p> $\log_{5\sqrt{5}} 125 = x \text{ (let)}$ $(5\sqrt{5})^x = 125$ <p>or, <math>(5\sqrt{5})^x = (5\sqrt{5})^2</math></p> <p>or, <math>x = 2</math></p>
61	(b)	$12! / (8! \times 4!) = 12 \times 11 \times 10 \times 9 \times 8! / (8! \times 4 \times 3 \times 2 \times 1)$ $= 12 \times 11 \times 10 \times 9 / 24 = 11880 / 24 = 495$
62	(c)	$(v-1)! \times 20 = (v+1) \times v \times (v-1)! \quad \text{Or, } 20 = (v+1) \times v$ <p>or, <math>5 \times 4 = (v+1) \times v</math> or, <math>v = 4</math>.</p>
63	(a)	$10C_1 \times 5C_1 = 10 \times 5 = 50$
64	(d)	$14! = 14 \times 13 \times 12! ; 12! = 1 \times 12! ; 13! = 13 \times 12! \quad \text{Hence, } 14!$
65	(a)	<p>First event : Burdwan to Haldia – 9 ways.</p> <p>Second Event : Haldia to Burdwan – (9-1) ways</p> <p>So, <math>9 \times 8</math> ways = 72 ways.</p>
66	(d)	${}^n P_6 : {}^n P_5 = 1:3 \quad \text{or, } \{n! / (n-6)!\} : \{n! / (n-5)!\} = 1:3$ <p>or, <math>n! / (n-6)! : n! / (n-5)(n-6)! = 1/3</math></p> <p>or, <math>n - 5 = 1/3</math> or, <math>n = 16/3</math>.</p>
67	(d)	$2/36! - 1/35! - 1/35! = 2/36! - 2/35! = 2/36! - (36 \times 2) / (36 \times 35!)$ $= 2/36! - 72/36! = (2-72) / 36! = -70/36!$
68	(a)	${}^{15} P_3 = 15! / (15-3)! = 15! / 12!$ $= 15 \times 14 \times 13 \times 12! / 12! = 15 \times 14 \times 13 = 2730.$
69	(a)	$(10! - 9!) / 8! = (10 \times 9 \times 8! - 9 \times 8!) / 8!$ $= 8!(10 \times 9 - 9) / 8! = 90 - 9 = 81$
70	(b)	<p>5 drivers can refill from 5 refills,</p> <p>Hence, possible ways : <math>5 \times 4 \times 3 \times 2 \times 1 = 120</math> ways.</p>
71	(a)	$8! / (8-8)! = 8! / 0! = 8! / 1 = 40320 \text{ ways}$
72	(b)	${}^{11} P_5 = 11! / (11-5)! = 11! / 6!$ $= 11 \times 10 \times 9 \times 8 \times 7 \times 6! / 6! = 11 \times 10 \times 9 \times 8 \times 7 = 55440$

73	(a)	$\frac{9}{7!} - \frac{3}{6!} + \frac{12}{7!} = \frac{(9+12)}{7!} - \frac{(7 \times 3)}{(7 \times 6!)}$ $= \frac{21}{7!} - \frac{21}{7!} = 0$
74	(c)	${}^n P_7 : {}^n P_8 = 1:4 \quad \text{or, } \{n!/(n-7)!\} : \{n!/(n-8)!\} = 1:4$ $\text{or, } \{n!/(n-7)(n-8)!\} \times (n-8)!/n! = 1/4$ $\text{or, } n-7 = 4 \quad \text{or, } n = 11.$
75	(d)	$(a-2)! \times 24 = (a+1)! \quad \text{Or, } (a-2)! \times 24 = (a+1) \times a \times (a-1) \times (a-2)!$ $\text{Or, } 24 = (a+1) \times a \times (a-1) \quad \text{or, } 4 \times 3 \times 2 = (a+1) \times a \times (a-1)$ $\text{or, } a = 3.$
76	(a)	$11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 / 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 11!/6!$
77	(a)	8 people and 8 vaccinators Hence, possible ways = $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$ ways
78	(c)	$4! / (4-4)! = 4!/0! = 4!/1 = 4! = 4 \times 3 \times 2 \times 1 = 24 \text{ ways}$
79	(a)	$\frac{3}{13!} - \frac{2}{12!} - \frac{2}{12!} = \frac{3}{13!} - \frac{4}{12!} = \frac{3}{13!} - \frac{(13 \times 4)}{(13 \times 12!)}$ $= \frac{3}{13!} - \frac{52}{13!} = \frac{(3-52)}{13!} = -\frac{49}{13!}$
80	(d)	$6!/(6-6)! = 6!/0! = 6!/1 = 6! = 720 \text{ ways}$
81	(c)	$b^2 - 4ac = (-24)^2 - 4(16)(9) = 576 - 576 = 0.$ Hence, Roots are Real and Equal.
82	(c)	Product of roots can be expressed as $c/a$ , Hence, $a = 1$ , product of roots = $c/a = c/1 = c$
83	(b)	Sum of the roots = $9 + (-4) = 5$ , Product of the Roots = $9 \times (-4) = -36$ $x^2 - (\text{Sum of the Roots}) \times x + (\text{Product of the Roots}) = 0$ $x^2 - (5)x + (-36) = x^2 - 5x - 36 = 0$
84	(a)	Sum of the Roots = $(a-t) + (a+t) = 2a$ , Product of the Roots = $(a-t) \times (a+t) = a^2 - t^2$ $x^2 - (\text{Sum of the Roots}) \times x + (\text{Product of the Roots}) = 0$ $x^2 - (2a)x + (a^2 - t^2) = x^2 - 2ax + a^2 - t^2 = 0$
85	(a)	$\alpha + \beta = -\frac{b}{a} = -\frac{(-7)}{5} = \frac{7}{5} \quad \text{and} \quad \alpha\beta = \frac{c}{a} = \frac{9}{5}. \text{ So } \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{\frac{7}{5}}{\frac{9}{5}} = \frac{7}{9}$
86	(b)	$(\alpha + \beta)^2 - 2\alpha\beta = (7/5)^2 - 2 \times (9/5) = 49/25 - 18/5 =$ $(49 - 90) / 25 = -41/25$

87	(d)	$\frac{7}{5} = \left(-\frac{b}{a}\right)$
88	(d)	$\frac{9}{5} = \left(\frac{c}{a}\right)$
89	(b)	$-41/45$
90	(a)	$(7/5) + 4 = (7 + 20)/5 = 27/5$
91	(d)	Real and Unequal
92	(c)	Product of roots can be expressed as $c/a$ , Hence, $a = 1$ , product of roots = $c/a = c/1 = c$
93	(a)	$ax^2 + bx + c = 0$
94	(d)	3 constants and a (coefficient of $x^2$ ) $\neq 0$ .
95	(c)	$b^2 - 4ac$ is known as Discriminant of the equation
96	(c)	For $b^2 - 4ac \geq 0$ , roots are real.
97	(b)	Sum of the roots is expressed as $-b/a$ , Hence, for $a = -1$ , Sum of the roots = $-b/(-1) = b$
98	(a)	Coefficient of $x$ ( $ax^2 + bx + c = 0$ )
99	(d)	$5x^2 - 60x + 115 = 0$ or, $x^2 - 12x + 23 = 0$ Sum ( $\alpha + \beta$ ): 12, Product ( $\alpha \times \beta$ ): 23, Square of Difference $(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$ $= (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta = (12)^2 - 4 \times (23) = 52$
100	(b)	Coefficient of $x^2$ ( $ax^2 + bx + c = 0$ )

• State True or False

1	2	3	4	5	6	7	8	9	10
T	T	F	T	F	T	F	F	T	T
11	12	13	14	15	16	17	18	19	20
F	T	F	F	T	F	T	T	T	F
21	22	23	24	25	26	27	28	29	
F	F	T	T	T	F	F	F	F	

● **Fill in the Blanks**

1	$\{1,3,6\}$	2	$\varphi$
3	$\{1,2,3\}$	4	$A = B$
5	$n(A) + n(B)$	6	2
7	Set of all integers except zero	8	$27abc$
9	$x - 1/x$	10	20
11	$1/4$	12	5
13	1000	14	16
15	16	16	12
17	31	18	14
19	$(-4)$	20	6
21	2	22	1
23	1023	24	252
25	1470	26	116
27	144	28	$(-21/4, -14/3)$
29	4	30	$(\pm 2\sqrt{5})$
31	$(-\frac{3}{4} \text{ and } -\frac{4}{3})$		

# CALCULUS – APPLICATION IN BUSINESS

# 3

**This Modules includes**

- 3.1 Concept of Calculus and its Application in Business**
- 3.2 Revenue and Cost Function**
- 3.3 Optimization Techniques (Basic Concepts)**

# CALCULUS – APPLICATION IN BUSINESS

## **Module Learning Objectives:**

After studying this Module, the students will be able to understand –

- ✦ The meaning of a Function
- ✦ The Concept of Limit of a Function
- ✦ The Concept of Differentiation of a Function
- ✦ The application of differentiation in business situations with special emphasis to the Cost and Revenue functions
- ✦ The application of differentiation for optimization

# Concept of Calculus and its Application in Business

## 3.1

### FUNCTION

If  $x$  and  $y$  be two real variables related to some rule, such that corresponding to every value of  $x$  within a defined domain or set of values we get a defined value of  $y$ , then  $y$  is said to be a function of  $x$  defined in its domain.

Here the variable  $x$  to which we may arbitrarily assign different values in the given domain is known as *Independent variable* (or argument) and  $y$  is called the *Dependent variable* (or function).

**Notations :** Generally we shall represent functions of  $x$  by the symbols  $f(x)$ ,  $F(x)$ ,  $y(x)$  etc.

**Illustration 1.** A man walks at an uniform rate of 5 km per hour. If  $s$  indicates the distances and  $t$  be the time in hours (from start), then we may write,  $s = 5t$ .

Here  $s$  and  $t$  are both variables,  $s$  is dependent if  $t$  is independent. Now  $s$  is a function of  $t$  and the domain (value) of  $t$  is  $0 \leq t \leq \infty$ .

**Illustration 2.**  $y = f(x) = \frac{x^2}{x}$ .

For  $x \neq 0$ ,  $y = x$  and for  $x = 0$ ,  $y$  is not known (undefined). Here the domain of  $x$  is the set of real numbers *except zero*. (refer worked out problem 2 of limit & continuity)

**Constant Function :**  $y = f(x) = 7$  for all real values of  $x$ . Here  $y$  has just one value 7 for all values of  $x$ .

**Single-valued, Multi-valued Function:** For a function sometimes it may so happen that for  $y = f(x)$ , there exists a single value of  $y$  for every value of  $x$ . This type of function is known as single-valued function.

**Illustration 3.**  $y = f(x) = 2x + 3$

For  $x = 1, y = 2.1 + 3 = 2 + 3 = 5$ .

$x = 2, y = 2.2 + 3 = 4 + 3 = 7$

If again we get more than one value of  $y$  for a value of  $x$ , then  $y$  is said to be a *multiple-valued* (or multi-valued) function of  $x$ .

**Illustration 4.**  $y^2 = x$ . Here for every  $x > 0$ , we find two values of  $y$  as  $y = \pm\sqrt{x}$

**Explicit and Implicit Function:** A function is said to be *explicit* when it is expressed directly in terms of the independent variable; otherwise it is implicit.

**Illustration 5.**  $y = x^2 - x + 1$  is an explicit function :

$2x^2 + 3xy + y^2 = 0$  an implicit function.

**Parametric Representation of a Function :** If the dependent variable  $x$  be expressed in terms of a third variable, say  $t$ , i.e.,  $y = f(t)$ ,  $x = F(t)$ , then these two relations together give the parametric representation of the function between  $y$  &  $x$ .

**Illustration 6.**  $y = t^2 + 1$ ,  $x = 2t$ .

**Odd and Even Functions :** A function  $f(x)$  is an odd function of  $x$  if  $f(-x) = -f(x)$  and is an even function of  $x$  if  $f(-x) = f(x)$ .

**Illustration 7.**  $f(x) = x$ . Now  $f(-x) = -x$  i.e  $f(-x) = -f(x)$ , so  $f(x) = x$  is an odd function of  $x$ .

$f(x) = x^2$ ,  $f(-x) = (-x)^2 = x^2 = f(x)$ , so  $f(x) = x^2$  is an even function of  $x$ .

**Inverse Function :** If from a function  $y = f(x)$ , we can obtain another function  $x = F(y)$ , then each function is known as the inverse of the other.

**Illustration 8.**  $y = 4x - 3$  and  $x = \frac{1}{4}(y + 3)$  are inverse to each other.

Both are the Functions of single independent variable :

**Polynomial Function :** A function of the form

$$F(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n,$$

(Where  $n$  is a positive integer and  $a_0, a_1, \dots, a_n$  are constants) is known as a polynomial function in  $x$ .

For  $n = 0$ ,  $f(x) = a_0$ , a constant function

$= 1$ ,  $f(x) = a_0 + a_1x$ , a linear function in  $x$

$= 2$ ,  $f(x) = a_0 + a_1x + a_2x^2$ , a quadratic function in  $x$ .

$= 3$ ,  $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ , a cubic function in  $x$ .

**Rational function :** A function that is expressed as the ratio of two polynomials

$$\text{i.e., } f(x) = \frac{a_0 + a_1x + a_2x^2 + \dots + a_nx^n}{b_0 + b_1x + b_2x^2 + \dots + b_nx^n}.$$

i.e., in the form of  $\frac{P(x)}{Q(x)}$

is called a rational function of  $x$ , such function exists for denominator  $\neq 0$ .

**Illustration 9.**  $f(x) = \frac{x+2}{x^2-4x+3}$  exists for all values of  $x$ , if  $x^2 - 4x + 3 \neq 0$ . Now for  $x^2 - 4x + 3 = 0$  or

$(x - 1)(x - 3) = 0$  or,  $x = 1, 3$  Denominator becomes zero and hence the given function does not exist.

**Irrational function :** On the contrary, if a function  $f(x)$  cannot be represented in this form, it is called an irrational function.

**Illustration 10.** Functions of the form  $\sqrt{x}$  (where  $x$  is not a perfect square number)

**Algebraic function :** A function in the form of a **polynomial** with **finite number of terms** is known as algebraic function.

**Illustration 11.**  $x^2 + 2x - 3, \sqrt{x^2 + 1}$  etc.

**Domain and Range of a Function :**

The set of values of independent variables  $x$  is called the 'Domain' of the function and the set of corresponding values of  $f(x)$  i.e. the dependent variable  $y$  is called the 'Range' of the function.

**Illustration 12.** For the squared function of  $y = x^2$ , we get the ordered pairs (1, 1) (2, 4) (3, 9), ... etc. as the values of  $x$  and  $y$  respectively, The set of values  $\{\dots -2, -1, 0, 1, 2, 3, \dots\}$  is the domain of the Independent Variable, where as set of values  $\{0, 1, 4, 9, \dots\}$  represents the range of the Dependent one.

**Illustration 13.** For the following functions find the domain and range.

$$(i) \quad f(x) = \frac{x^2 - 4}{x - 2}, x \neq 2 \quad (ii) \quad f(x) = \frac{3 - x}{x - 3}, \quad (iii) \quad f(x) = \frac{2x + 1}{(x - 1)(x + 1)}$$

**Solution:**

$$(i) \quad f(0) = \frac{-4}{-2} = 2, f(1) = \frac{-3}{-1} = 3, f(3) = \frac{5}{1} = 5, f(4) = 6, f(-1) = 1$$

$$\therefore \text{domain} = \{\dots -2, -1, 0, 1, 3, 4, \dots\}, \quad \text{range} = \{1, 2, 3, 5, 6, \dots\}$$

$$= \mathbb{R} - \{2\},$$

$$= \mathbb{R} - \{4\}, \mathbb{R} = \text{real number},$$

$$(ii) \quad f(0) = \frac{3}{-3} = -1, f(1) = -1, f(2) = -1, f(-1) = -1$$

$$\text{domain} = \{-1, 0, 1, 2, 4, \dots\}, \quad \text{range} = \{-1, -1, -1, \dots\} = \{-1\}$$

$$= \mathbb{R} - \{3\}, \text{ where } \mathbb{R} \text{ is a real number}$$

(iii) The function of  $f(x)$  exists for  $x \neq 1, x \neq -1$ .

$$\therefore \text{domain} = \mathbb{R} - \{-1, 1\}, \text{ i.e. all real numbers excluding } 1 \text{ \& } (-1)$$

$$\text{range} = \mathbb{R}.$$

**Illustration 14.** Find the domain of definition of the function  $\frac{4x - 5}{\sqrt{x^2 - 7x + 12}}$

**Solution:**

$$\text{Denominator} = \sqrt{x^2 - 7x + 12} = \sqrt{x^2 - 4x - 3x + 12} = \sqrt{x(x - 4) - 3(x - 4)} = \sqrt{(x - 3)(x - 4)}$$

From above it can be said that if  $3 < x < 4$  then  $(x - 3)$  is positive and  $(x - 4)$  is negative.

Thus  $(x - 3)(x - 4) = (\text{Positive}) \times (\text{Negative}) = \text{Negative}$

Hence  $\sqrt{(x - 3)(x - 4)} = \sqrt{\text{Negative}} = \text{Imaginary number}$ . Also for  $x = 3$  &  $4$ , denominator is zero.

Thus given function is not defined for  $3 \leq x \leq 4$ . So domain is all real values of  $x$  except  $3 \leq x \leq 4$ .

**Absolute Value :** A real number "a" may be either  $a = 0$  or,  $a > 0$  or  $a < 0$ . The absolute value (or modulus) of a, denoted by  $|a|$  is defined as  $|a| = a$ , for  $a > 0$

$$= -a, \text{ for } a < 0$$

Thus  $|-4| = -(-4) = 4$ , and  $|4| = 4$ .

**Complex No :**

A number of the form  $(a+ib)$  or  $(a-ib)$  [where  $a$  &  $b$  are real numbers] is called a complex number (where  $i = \sqrt{-1}$ )

The complex number has two parts; a real part & an imaginary part. 'a' is the real part & 'ib' is the imaginary part.

**Illustration 15.**

If  $y = x^2 + 4$  is a function under consideration then solving for  $y = 0$ , we get

$$0 = x^2 + 4$$

$$\text{or, } x^2 = -4$$

$$\text{or, } x = \pm\sqrt{-4}$$

$$\text{or, } x = \pm 2\sqrt{-1} = \pm 2i$$

The number  $\pm 2i$  is a complex number whose real part is 0 & imaginary part is  $\pm 2$

**Illustration 16.** Given  $f(x) = 2x^2 - 3x + 1$ ; find  $f(2)$ ,  $f(0)$ ,  $f(-3)$

**Solution:**

$$f(2) = 2.2^2 - 3.2 + 1 = 2.4 - 6 + 1 = 8 - 6 + 1 = 3$$

$$f(0) = 2.0^2 - 3.0 + 1 = 2.0 - 0 + 1 = 0 - 0 + 1 = 1$$

$$f(-3) = 2(-3)^2 - 3(-3) + 1 = 2.9 + 9 + 1 = 18 + 9 + 1 = 28.$$

**Illustration 17.** If  $y = 4x - 1$ , find the value of  $y$  for  $x = 2$ . Can  $y$  be regarded as a function of  $x$ ? Also find the domain.

**Solution:**

For  $x = 2$ ,  $y = 4.2 - 1 = 8 - 1 = 7$ . Again for  $x = 0$ ,  $y = -1$  and for  $x = -1$ ,  $y = -5$ . So for every value of  $x$  in  $-\infty < x < \infty$ , we find different values of  $y$ , So  $y$  is a function of  $x$  and its domain is -

$$-\infty < x < +\infty.$$

**Illustration 18.** If  $f(x) = x + |x|$ , find  $f(3)$  and  $f(-3)$  and show also they are not equal.

**Solution:**

$$f(3) = 3 + |3| = 3 + 3 = 6; f(-3) = -3 + |-3| = -3 + 3 = 0.$$

As  $6 \neq 0$ , so  $f(3) \neq f(-3)$ .

**Note :** If  $f(x) = f(-x)$  [i.e.,  $f(3) = f(-3)$ ] then  $f(x)$  will be an even function of  $x$ .

**Illustration 19.** Show that  $\sqrt{x^2 - 5x + 4}$  is not defined for  $1 < x < 4$

**Solution:**

$$\sqrt{x^2 - 5x + 4} = \sqrt{(x-1)(x-4)}$$

Now for any value  $x > 1$ , but  $< 4$  the expression becomes imaginary. So the expression is undefined for  $1 < x < 4$ .

**Illustration 20.** Find the domain of  $f(x) = \frac{x}{x^2 - 9}$

**Solution:**

Here  $f(x)$  has a unique value except for  $x = 3, -3$ .

$$\text{For } f(3) = \frac{3}{9-9} = \frac{3}{0} \text{ (undefined) and } f(-3) = \frac{-3}{9-9} = \frac{-3}{0} \text{ (undefined)}$$

$\therefore$  domain of the function  $f(x)$  is  $-\infty < x < -3$ ;  $-3 < x < 3$  and  $3 < x < \infty$ .

**Illustration 21.** Given the function

$$\begin{aligned} f(x) &= 5^{-2x} - 1, -1 \leq x < 0 \\ &= \frac{x^2 - 2}{x - 2}, 0 \leq x < 1 \\ &= \frac{2x}{x^2 - 1}, 1 \leq x < 3 \end{aligned}$$

Find  $f(-1)$ ,  $f(0)$ ,  $f(1/2)$ ,  $f(2)$ .

**Solution:**

$$\begin{aligned} f(-1) &= 5^{-2(-1)} - 1 \text{ (since } -1 \text{ lies in the interval } -1 \leq x < 0) \\ &= 5^2 - 1 = 25 - 1 = 24. \end{aligned}$$

$x = 0$  and  $x = \frac{1}{2}$  lie in the interval  $0 \leq x < 1$ ,

$$f(0) = \frac{0-2}{0-2} = 1, \quad f\left(\frac{1}{2}\right) = \frac{\frac{1}{4}-2}{\frac{1}{2}-2} = \frac{7}{6}.$$

Now 2 lies in the third interval. So  $f(2) = \frac{2 \cdot 2}{2^2 - 1} = \frac{4}{4 - 1} = \frac{4}{3}$ .

**Illustration 22.** If  $f(x) = e^{ax+b}$ . Prove that  $e^b f(x+y) = f(x) \cdot f(y)$

**Solution:**

$$\begin{aligned} e^b f(x+y) &= e^b \cdot e^{a(x+y)+b} = e^{b+ax+ay+b} \\ &= e^{ax+b} \cdot e^{ay+b} = f(x) \cdot f(y). \text{ Hence proved.} \end{aligned}$$

**Illustration 23.** If  $f(x) = x - a$ ,  $q(x) = x + a$  then show that

$$\{f(x)\}^2 - \{q(x)\}^2 = -2a \{f(x) + q(x)\}$$

**Solution:**

$$\text{L.H.S.} = (x - a)^2 - (x + a)^2 = x^2 + a^2 - 2ax - (x^2 + a^2 + 2ax) = -4ax$$

$$\text{R.H.S.} = -2a \{x - a + x + a\} = -2a \cdot 2x = -4ax. \text{ So L.H.S.} = \text{R.H.S. (Proved)}$$

## LIMIT

**Introduction:**

Calculus is based, in general, on the idea of limit. At present this idea including its related concepts, continuity to mention, will be discussed.

**Some definitions:**

- (i) **Meaning of “x tends to a”.** When the difference  $|x-a|$  (i.e., numerical difference between the present value of  $x$  and  $a$ ) can be made less than any positive quantity, however small, we say  $x$  tends to  $a$  and is written as  $x \rightarrow a$ .
- (ii) **Meaning of “x tends to zero”.** When the value of  $x$  goes on decreasing numerically and can be made numerically less than any positive quantity, however small, we say  $x$  tends to zero and is written as  $x \rightarrow 0$ .
- (iii) **Meaning of “x tends to infinity”.** When the value of  $x$  goes on increasing and can be made greater than any positive quantity, however, large, we say  $x$  tends to infinity written as  $x \rightarrow \infty$ .

**Neighbourhood or Proximity of a point**

Let  $c$  be any real number, then any open interval around  $c$  is called the neighbourhood of  $c$ , e.g.,

$$] c - \epsilon, c + \epsilon[, \epsilon > 0$$

Is the neighbourhood of  $c$ .

Any neighbourhood from which the point  $c$  is excluded is called deleted neighbourhood of  $c$ .

Geometrically it means set of those points which are within an infinitely small distance  $\epsilon$  from  $c$  on either side except for the point  $c$ .

(iv) **Limit of a function  $f(x)$ .** A number  $l$  is said to be the limit of  $f(x)$  as  $x \rightarrow a$  written as  $\lim_{x \rightarrow a} f(x) = l$  if

- a. The function is defined and single valued in the deleted neighbourhood of  $a$ .
- b. For every positive number  $\epsilon$ , however small, there exists a positive number  $\delta$  (usually depending on  $\epsilon$ ), such

$$|f(x) - l| < \epsilon$$

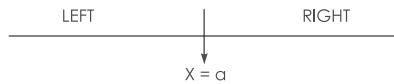
Whenever  $0 < |x - a| < \delta$

From the above definition it is interesting to note  $\lim_{x \rightarrow a} f(x)$  that may exist, even if the function  $f(x)$  is not defined at  $x = a$ . Sometimes both the things may happen, i.e.,

(i) The function is defined at  $x = a$ , and

(ii)  $\lim_{x \rightarrow a} f(x)$  also exists.

**Right hand and left hand limits**



The variable point  $x$  can approach ‘ $a$ ’ either from the left or from the right. These respective approaches are indicated by writing  $x \rightarrow a^-$  and  $x \rightarrow a^+$

When

$$\lim_{x \rightarrow a^-} f(x) = l_1 \text{ and } \lim_{x \rightarrow a^+} f(x) = l_2 \text{ then we can say } \lim_{x \rightarrow a} f(x) = l \text{ if and only if } l_1 = l_2 = l$$

**Methods of finding limit of a function  $f(x)$  as  $x$  tends to a finite quantity say ‘ $a$ ’**

There are three methods for finding limit of a function  $f(x)$  as  $x$  tends to a finite quantity say ‘ $a$ ’:

- (i) Method of factors
- (ii) Method of substitution
- (iii) Method of rationalization.

In method I, if  $f(x)$  is of the form  $\frac{g(x)}{h(x)}$  factorise  $g(x)$  and  $h(x)$ , cancel the common factors and then put the value of  $x$ .

**Illustration 24.** Find the value of  $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1}$ .

**Solution:**

$$\begin{aligned}\text{Now, } \frac{x^3 - 1}{x^2 - 1} &= \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} \\ &= \frac{x^2 + x + 1}{x+1} \quad [\text{As } x \rightarrow 1, x \neq 1 \text{ i.e. } x-1 \neq 0] \\ \therefore \lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1} &= \lim_{x \rightarrow 1} \frac{x^2 + x + 1}{x+1} \\ &= \frac{1+1+1}{1+1} = \frac{3}{2}.\end{aligned}$$

**Method II**

The following steps are involved:

- (i) Put  $x = a + h$  where  $h$  is very small but  $\neq 0$ , i.e.  $x \rightarrow a, h \rightarrow 0$ .
- (ii) Simplify numerator and denominator and cancel common powers of  $h$ .
- (iii) Put  $h = 0$ .

The result is the required limit.

**Illustration 25.** Evaluate  $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}$ .

**Solution:**

Put  $x = a + h$  where  $h$  is very small, then

$$\begin{aligned}\frac{x^n - a^n}{x - a} &= \frac{(a+h)^n - a^n}{h} \\ &= \frac{a^n \left[ \left(1 + \frac{h}{a}\right)^n - 1 \right]}{h} \\ &= \frac{a^n \left[ \left(1 + \frac{nh}{a} + n(n-1)\frac{h^2}{a^2} + \dots\right) - 1 \right]}{h} \\ &= \frac{a^n \left[ n\frac{h}{a} + n(n-1)\frac{h^2}{a^2} + \dots \right]}{h} \\ &= a^n \left[ \frac{n}{a} + n(n-1)\frac{h}{a^2} + \dots \right] \\ \therefore \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{h \rightarrow 0} a^n \left[ \frac{n}{a} + n(n-1)\frac{h}{a^2} + \dots \right] \\ &= a^n \cdot \frac{n}{a} = na^{n-1}\end{aligned}$$

**Method III. (Rationalisation)**

This method is useful where radical signs are involved either in the numerator or denominator. The numerator or

denominator (as required) is rationalized and limit taken. The following **Illustration** will make the method clear.

**Illustration 26.** Evaluate  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

**Solution:**

Rationalising the numerator, we get

$$\begin{aligned} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} &= \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \\ &= \frac{2x}{x[\sqrt{1+x} + \sqrt{1-x}]} \\ &= \frac{2}{\sqrt{1+x} + \sqrt{1-x}} \quad [\text{As } x \rightarrow 0, x \neq 0] \\ \therefore \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} &= \lim_{x \rightarrow 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} = 1 \end{aligned}$$

### Infinite Limits

For finding the limit of  $f(x) = \frac{g(x)}{h(x)}$  as  $x \rightarrow \infty$ , we divide the numerator and denominator by highest power of  $x$  occurring in  $f(x)$  (numerator or denominator whichever has higher power of  $x$ ) and then use  $\frac{1}{x}$ ,  $\frac{1}{x^2}$ , etc.  $\rightarrow 0$  as  $x \rightarrow \infty$ .

**Illustration 27.** Evaluate

$$\lim_{x \rightarrow \infty} \frac{(x+1)(2x+3)}{(x+2)(3x+4)}$$

**Solution:**

$$\begin{aligned} \text{Now } \frac{(x+1)(2x+3)}{(x+2)(3x+4)} &= \frac{2x^2 + 5x + 3}{3x^2 + 10x + 8} \\ &= \frac{2 + \frac{5}{x} + \frac{3}{x^2}}{3 + \frac{10}{x} + \frac{8}{x^2}} \quad [\text{Dividing by } x^2] \end{aligned}$$

$$\therefore \lim_{x \rightarrow \infty} \frac{(x+1)(2x+3)}{(x+2)(3x+4)} = \frac{2+5 \times 0+3 \times 0}{3+10 \times 0+8 \times 0} \quad [\text{As } x \rightarrow \infty, \frac{1}{x} \text{ as well as } \frac{1}{x^2} \rightarrow 0] = \frac{2}{3}$$

**Alternative approach:**

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{(x+1)(2x+3)}{(x+2)(3x+4)} &= \lim_{x \rightarrow \infty} \frac{\left(1 + \frac{1}{x}\right) \left(2 + \frac{3}{x}\right)}{\left(1 + \frac{2}{x}\right) \left(3 + \frac{4}{x}\right)} = \frac{1 \cdot 2}{1 \cdot 3} = \frac{2}{3} \end{aligned}$$

**Illustration 28.**  $\text{Lt}_{x \rightarrow a} \frac{x\sqrt{x} - a\sqrt{a}}{x - a}$ .

**Solution:**

$$\text{Lt}_{x \rightarrow a} \frac{x^{\frac{3}{2}} - a^{\frac{3}{2}}}{x - a} = \frac{3}{2} a^{\frac{1}{2}}.$$

**Illustration 29.** If  $\text{Lt}_{(x \rightarrow 2)} \frac{x^n - 2^n}{x - 2} = 80$ , find 'n'

**Solution:**

$$n2^{n-1} = 80$$

Or  $n2^n = 160 = 5 \cdot 2^5$

Or  $n = 5$

**Illustration 30.**  $\text{Lt}_{(x \rightarrow 0)} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$ .

**Solution:**

$$\frac{\text{Lt}_{(x \rightarrow 0)} \frac{(1+x)^6 - 1}{x}}{\text{Lt}_{(x \rightarrow 0)} \frac{(1+x)^2 - 1}{x}} = \frac{6}{2} = 3.$$

## SOLVED ILLUSTRATIONS

**Illustration 31.**  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$ .

For if  $x = 2 + h$ , whether  $h$  be positive or negative,

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{(x - 2)} = \frac{h(4 + h)}{h} = 4 + h$$

and by taking  $h$  numerically small, the difference of  $\frac{x^2 - 4}{x - 2}$  and 4 can be made as small as we like. It may be noted here that however small  $h$  may be, as  $h \neq 0$ , one can cancel the factor  $(x - 2)$  i.e.,  $h$  between numerator and denominator here. Hence  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$ . But for  $x = 2$ , the function  $\frac{x^2 - 4}{x - 2}$  is undefined as we cannot cancel the factor  $x - 2$ , which is equal to zero.

Now writing  $f(x) = \frac{x^2 - 4}{x - 2}$ ,  $\lim_{x \rightarrow 2} f(x) = 4$  whereas  $f(2)$  does not exist.

**Illustration 32.** For  $f(x) = |x|$ , find,  $\lim_{x \rightarrow 0} f(x)$ .

**Solution:**

$$f(x) = |x| \text{ means } f(x) = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$$

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$  [ $x \rightarrow 0^+$  means  $x$  is approaching 0 from right i.e  $x > 0$  now. Also  $|x| = x$  for  $x > 0$ ]

$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0$ . [ $x \rightarrow 0^-$  means  $x$  is approaching 0 from left i.e  $x < 0$  now. Also  $|x| = -x$  for  $x < 0$ ]

So  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = 0$

$\therefore \lim_{x \rightarrow 0} f(x) = 0$ .

**Illustration 33.** Find analytically  $\lim_{x \rightarrow 3} \sqrt{x-3}$ , if it exists.

**Solution:**

$\lim_{x \rightarrow 3^+} \sqrt{x-3} = 0$ , but  $\lim_{x \rightarrow 3^-} \sqrt{x-3}$  does not exist. Because when  $x \rightarrow 3^-$  then  $x$  is approaching 3 from left which means  $x$  is definitely less than 3. Thus  $(x-3)$  is negative and square root of negative quantity is never real but imaginary.

$\therefore$  the limit does not exist.

**Alternative approach:**

Put  $x = 3 + h$ , as  $x \rightarrow 3$ ,  $h \rightarrow 0$

$$\lim_{x \rightarrow 3} \sqrt{x-3} = \lim_{h \rightarrow 0} \sqrt{3+h-3} = \lim_{h \rightarrow 0} \sqrt{h}$$

$\lim_{h \rightarrow 0^+} \sqrt{h} = 0$ , but  $\lim_{h \rightarrow 0^-} \sqrt{h}$  does not exist (as the  $h$  near to but less than zero corresponds no real value of  $\sqrt{h}$ ).

$\therefore \lim_{x \rightarrow 3} \sqrt{x-3}$  does not exist.

**Note :** At  $x = 3$ ,  $f(x) = \sqrt{x-3} = \sqrt{3-3} = 0 \therefore f(3)$  exists.

**Illustration 34.** Do the following limits exist? If so find the values

(i)  $\lim_{x \rightarrow -2} \frac{1}{x+2}$       (ii)  $\lim_{x \rightarrow 0} \frac{1}{x}$       (iii)  $\lim_{x \rightarrow 1} \left\{ (x^2 - 1) + \frac{(x-1)^2}{x-1} \right\}$

**Solution:**

(i)  $\lim_{x \rightarrow -2} \frac{1}{x+2} = \lim_{h \rightarrow 0} \frac{1}{-2+h+2} = \lim_{h \rightarrow 0} \frac{1}{h}$

Now  $\lim_{x \rightarrow 0^+} \frac{1}{h} = +ve$ ;  $\lim_{h \rightarrow 0^-} \frac{1}{h} = -ve$ . As the two limits are not same, so the limit does not exist.

(ii)  $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$ ,  $\lim_{h \rightarrow 0^-} \frac{1}{x} = -\infty$ . The limit doesn't exist as the two values are unequal.

$$\begin{aligned} \text{(iii)} \quad \lim_{x \rightarrow 1} \left\{ (x^2 - 1) + \frac{(x-1)(x-1)}{(x-1)} \right\} &= \lim_{x \rightarrow 1} \{ (x^2 - 1) + (x-1) \}, [\text{As } x \rightarrow 1, x \neq 1 \text{ i.e. } x-1 \neq 0] \\ &= \lim_{x \rightarrow 1} (x^2 + x - 2) = (1+1-2) = 0. \end{aligned}$$

On putting the limiting value of  $x$ , the value of the function exists and its value is 0.

### Distinction between $\lim_{x \rightarrow a} f(x)$ and $f(a)$

By  $\lim_{x \rightarrow a} f(x)$  we mean the value of  $f(x)$  when  $x$  has any arbitrary value very near 'a' but not 'a'. The quantity  $f(a)$  is the value of  $f(x)$ , when  $x$  is exactly equal to 'a'.

**Note :** The following cases may arise :

(i)  $f(a)$  does not exist, but  $\lim_{x \rightarrow a} f(x)$  exists.

Such situation is explained in example 28 above.

(ii)  $f(a)$  exists, but  $\lim_{x \rightarrow a} f(x)$  does not exist.

Such situation is explained in example 30 above.

(iii)  $f(a)$  and both exist,  $\lim_{x \rightarrow a} f(x)$  both exist but unequal.

$$\begin{aligned} \text{Let } f(x) &= 0, & \text{for } x \neq 0 \\ &= 1, & \text{for } x = 0 \end{aligned}$$

$$\lim_{x \rightarrow 0^+} f(x) = 0 = \lim_{x \rightarrow 0^-} f(x) \text{ and } f(0) = 1, \text{ unequal values.}$$

(iv)  $f(a)$  and  $\lim_{x \rightarrow a} f(x)$  both exist and equal.

Such situation is explained in example 29 above.

(v) neither  $f(a)$  nor,  $\lim_{x \rightarrow a} f(x)$  exists.

### Fundamental Theorem on Limits:

If  $\lim_{x \rightarrow a} f(x) = l$  and  $\lim_{x \rightarrow a} \phi(x) = m$ , where  $l$  and  $m$  are finite quantities then

$$1. \quad \lim_{x \rightarrow a} [f(x) \pm \phi(x)] = l \pm m$$

$$2. \quad \lim_{x \rightarrow a} [f(x) \cdot \phi(x)] = lm.$$

$$3. \quad \lim_{x \rightarrow a} \frac{f(x)}{\phi(x)} = \frac{l}{m}, m \neq 0$$

4. If  $\lim_{x \rightarrow a} \phi(x) = b$  and  $\lim_{x \rightarrow b} f(u) = f(b)$  then

$$\lim_{x \rightarrow b} f\{\phi(x)\} = f\left\{\lim_{x \rightarrow b} \phi(x)\right\} = f(b).$$

5.  $\lim_{x \rightarrow a} k = k$ , where  $k = \text{Constant}$
6.  $\lim_{x \rightarrow a} k.f(x) = k \lim_{x \rightarrow a} f(x)$ , where  $k = \text{Constant}$

**Illustration 35.** Evaluate,  $\lim_{x \rightarrow 1} \frac{x^2 + 3x - 1}{2x + 4}$ .

**Solution:**

As the limit of the denominator  $\neq 0$ , we get

$$\begin{aligned} \text{Required limit} &= \frac{\lim_{x \rightarrow 1} (x^2 + 3x - 1)}{\lim_{x \rightarrow 1} (2x + 4)} = \frac{\lim_{x \rightarrow 1} x^2 + \lim_{x \rightarrow 1} 3x - \lim_{x \rightarrow 1} 1}{\lim_{x \rightarrow 1} 2x + \lim_{x \rightarrow 1} 4} \quad (\text{by theorem 1}) \\ &= \frac{1 + 3 \cdot 1 - 1}{2 \cdot 1 + 4} = \frac{3}{6} = \frac{1}{2} \end{aligned}$$

We have not applied the definition to save labour. If we substitute  $x = 1$ , we get the value of the function  $= \frac{1}{2}$  (equal to the limit the value as  $x \rightarrow 1$ ). Practically this may not happen always, as shown below.

**Illustration 36.** Evaluate  $\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 4x + 3}$ .

$$\begin{aligned} \text{Required limit} &= \lim_{x \rightarrow 1} \frac{(x-1)(x-2)}{(x-1)(x-3)} = \lim_{x \rightarrow 1} \frac{x-2}{x-3} \quad [\text{As } x \rightarrow 1, x \neq 1 \text{ i.e. } x-1 \neq 0] \\ &= \frac{1-2}{1-3} = \frac{1}{2}. \end{aligned}$$

$$\lim_{x \rightarrow 1} \frac{(x-1)(x-2)}{(x-1)(x-3)} = \lim_{x \rightarrow 1} \frac{x-2}{x-3} = \frac{1-2}{1-3} = \frac{1}{2}.$$

If we put  $x = 1$ , the function becomes  $\frac{0}{0}$  which is undefined. Further limit of the denominator is zero, we cannot apply theorem 2, hence cancelling the common factor  $(x - 1)$  which is  $\neq 0$  as  $x \rightarrow 1$ , we obtain the above result.

**Illustrations related to variable tends of infinity:**

**Illustration 37.** Find the limit of  $\lim_{x \rightarrow \infty} (2x^2 - 5x + 2)$

**Solution:**

$$\text{Now } 2x^2 - 5x + 2 = x^2 \left( 2 - \frac{5}{x} + \frac{2}{x^2} \right)$$

$$\therefore \lim_{x \rightarrow \infty} (2x^2 - 5x + 2) = \lim_{x \rightarrow \infty} x^2 \left( 2 - \frac{5}{x} + \frac{2}{x^2} \right) = \lim_{x \rightarrow \infty} x^2 \times \lim_{x \rightarrow \infty} \left( 2 - \frac{5}{x} + \frac{2}{x^2} \right)$$

$$\text{Now } \lim_{x \rightarrow \infty} x^2 = \infty; \lim_{x \rightarrow \infty} 2 = 2; \lim_{x \rightarrow \infty} \frac{5}{x} = 0; \lim_{x \rightarrow \infty} \frac{2}{x^2} = 0 \text{ [As } x \rightarrow \infty, \frac{1}{x} \text{ \& } \frac{1}{x^2} = 0]$$

$$\therefore \lim_{x \rightarrow \infty} (2x^2 - 5x + 2) = \infty \times 2 = \infty.$$

**Illustration 38.** Find  $\lim_{x \rightarrow \infty} \frac{4x^5 + 2x^3 - 5}{7x^8 + x^4 + 2}$

**Solution:**

Of all the terms in numerator and denominator the highest power of  $x$  is 8. We now divide both the numerator and the denominator by  $x^8$  to avoid the undefined form  $\frac{\infty}{\infty}$ . So we get

$$\text{Given limit} = \lim_{x \rightarrow \infty} \frac{\frac{4}{x^3} + \frac{2}{x^5} - \frac{5}{x^8}}{7 + \frac{1}{x^4} + \frac{2}{x^8}} = \lim_{u \rightarrow 0} \frac{4u^3 + 2u^5 - 5u^8}{7 + u^4 + 2u^8} \text{ (put } \frac{1}{x} = u, \text{ so as } x \rightarrow \infty, u \rightarrow 0)$$

$$= \frac{\lim_{u \rightarrow 0} (4u^3 + 2u^5 - 5u^8)}{\lim_{u \rightarrow 0} (7 + u^4 + 2u^8)}$$

$$= \frac{4 \lim_{u \rightarrow 0} u^3 + 2 \lim_{u \rightarrow 0} u^5 - 5 \lim_{u \rightarrow 0} u^8}{7 + \lim_{u \rightarrow 0} u^4 + 2 \lim_{u \rightarrow 0} u^8} = \frac{0 + 0 - 0}{7 + 0 + 0} = \frac{0}{7} = 0$$

**Illustration 39.** Find  $\lim_{x \rightarrow \infty} \frac{5 - 2x^2}{3x + 5x^2}$ .

**Solution:**

$$\text{Given limit} = \lim_{x \rightarrow \infty} \frac{\frac{5}{x^2} - 2}{\frac{3}{x} + 5} = \lim_{u \rightarrow 0} \frac{5u^2 - 2}{3u + 5} \text{ (Let } \frac{1}{x} = u, \text{ as } x \rightarrow \infty, u \rightarrow 0)$$

$$= \frac{5 \lim_{u \rightarrow 0} u^2 - 2}{3 \lim_{u \rightarrow 0} u + 5} = \frac{0 - 2}{0 + 5} = -\frac{2}{5}.$$

**Illustration related to rationalisation:**

**Illustration 40.** Find the value of :  $\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

$$\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}, \text{ [As } h \rightarrow 0, h \neq 0 \text{]} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

**Illustrations related to left hand and right hand limits:**

**Illustration 41.** A function is defined as

$$\begin{aligned} f(x) &= x^2, & \text{for } x > 1 \\ &= 4.1, & \text{for } x = 1 \\ &= 2x, & \text{for } x < 1. \end{aligned}$$

Does  $\lim_{x \rightarrow 1} f(x)$  exist?

**Solution:**

We have to find R.H. limit and L.H. limit i.e  $\lim_{x \rightarrow 1+} f(x)$  and  $\lim_{x \rightarrow 1-} f(x)$ .

Now  $\lim_{x \rightarrow 1+} f(x) = \lim_{x \rightarrow 1+} x^2 = 1^2 = 1$ , [As  $f(x) = x^2$  for  $x > 1$ ]

Again  $\lim_{x \rightarrow 1-} f(x) = \lim_{x \rightarrow 1-} 2x = 2 \times 1 = 2$ , [As  $f(x) = 2x$  for  $x < 1$ ]

Since the values of R.H. L and L.H.L are not equal, so  $\lim_{x \rightarrow 1} f(x)$  doesn't exist.

**Illustration 42.**

Evaluate (i)  $\lim_{x \rightarrow 0} \frac{4x + |x|}{3x + |x|}$  (ii)  $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$

**Solution:**

(i) R.H. Limit =  $\lim_{x \rightarrow 0+} \frac{4x + x}{3x + x} = \lim_{x \rightarrow 0} \frac{5x}{4x} = \frac{5}{4}$ , [As  $x \rightarrow 0$ , So  $x \neq 0$  and also for  $x > 0$ ,  $|x| = x$ ]

L.H. Limit =  $\lim_{x \rightarrow 0-} \frac{4x - x}{3x - x} = \lim_{x \rightarrow 0-} \frac{3x}{2x} = \frac{3}{2}$  [As  $x \rightarrow 0$ , So  $x \neq 0$ , and for  $x < 0$ ,  $|x| = -x$ ]

∴ the given limit doesn't exist, as the values of R.H.L and L.H.L are unequal.

(ii) R.H.L =  $\lim_{x \rightarrow 2+} \frac{|x-2|}{x-2} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)} = 1$ . [As  $x \rightarrow 2$ ,  $x \neq 2$  i.e  $(x-2) \neq 0$ . Also for  $|x-2| > 0$  i.e., for  $x > 2$ ,

$$|x-2| = x-2$$

L.H.L =  $\lim_{x \rightarrow 2-} \frac{|x-2|}{x-2} = \lim_{x \rightarrow 2-} \frac{-(x-2)}{(x-2)} = -1$  for  $|x-2| < 0$  i.e., for  $x < 2$ ,  $|x-2| = -(x-2)$

As the values of R.H.L and L.H.L are unequal, so the given limit does not exist.

**Some Useful Limits :**

(A)  $\lim_{x \rightarrow 1} (1+x)^{1/x} = e$

(C)  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

(E)  $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(B)  $\lim_{x \rightarrow 0} \frac{1}{x} \log_e (1+x) = 1$

(D)  $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$

(F)  $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x} = n.$

**Illustration 43.** Evaluate  $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x}$ **Solution:**Put  $3x = u$ , As  $x \rightarrow 0$   $u \rightarrow 0$ .

$$\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x} = \lim_{u \rightarrow 0} \frac{e^u - 1}{u/3} = 3 \times \lim_{u \rightarrow 0} \frac{e^u - 1}{u} = 3 \times 1 \text{ (by C)} = 3.$$

**Illustration 44.** Evaluate  $\lim_{x \rightarrow 0} \frac{\log(1+6x)}{x}$ **Solution:**Put  $6x = u$ , As  $x \rightarrow 0$ ,  $u \rightarrow 0$ 

$$\text{Given limit} = \lim_{u \rightarrow 0} \frac{\log(1+u)}{u/6} = 6 \times \lim_{u \rightarrow 0} \frac{\log(1+u)}{u} = 6 \times 1 \text{ (by B)} = 6.$$

**Illustration 45.**  $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$ **Solution:**

$$\begin{aligned} \text{Given limit} &= \lim_{x \rightarrow 0} \frac{a^x - 1 - b^x + 1}{x} = \lim_{x \rightarrow 0} \frac{a^x - 1}{x} - \lim_{x \rightarrow 0} \frac{b^x - 1}{x} \\ &= \log a - \log b \text{ [by (D) above]} = \log \frac{a}{b}. \end{aligned}$$

**Illustration 46.**  $\lim_{x \rightarrow 0} \frac{e^{ax} - e^{bx}}{x}$ **Solution:**

$$\begin{aligned} \text{Given limit} &= \lim_{x \rightarrow 0} \frac{(e^{ax} - 1) - (e^{bx} - 1)}{x} = \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{x} - \lim_{x \rightarrow 0} \frac{e^{bx} - 1}{x} \\ &= \lim_{u \rightarrow 0} \frac{e^u - 1}{u/a} - \lim_{v \rightarrow 0} \frac{e^v - 1}{v/b} \text{ [Let } ax = u, \text{ As } x \rightarrow 0 \text{ } u \rightarrow 0 \text{ also. Similarly for } bx = v, \text{ As } x \rightarrow 0, v \rightarrow 0 \text{ also]} \\ &= a \lim_{u \rightarrow 0} \frac{e^u - 1}{u} - b \lim_{v \rightarrow 0} \frac{e^v - 1}{v} \\ &= a \cdot 1 - b \cdot 1 = a - b. \end{aligned}$$

## CONTINUITY

### Introduction :

A function is said to be continuous if its graph is a continuous curve without any break. If, however, there is any break in the graph, then function is not continuous at that point.

If for a value of  $k$ , the limit of  $f(x)$  does not exist i.e., if on the curve of  $f(x)$  a point is absent, the graph will be discontinuous i.e., not continuous.

A function  $f(x)$  is said to be continuous at  $x = a$ , when  $\lim_{x \rightarrow a} f(x)$  if exists is finite and is equal to  $f(a)$

$$\text{i.e., } \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a)$$

$$\text{i.e., } f(a + 0) = f(a - 0) = f(a) \text{ briefly or } \lim_{x \rightarrow 0} f(a + h) = f(a)$$

Thus we are to find following three values :

$$(i) \lim_{x \rightarrow a^+} f(x), \quad (ii) \lim_{x \rightarrow a^-} f(x) \quad (iii) f(a)$$

If however all these values are equal, then  $f(x)$  is continuous at  $x = a$ , otherwise it is discontinuous.

**Illustration 47.** Show that  $f(x) = 3x^2 - x + 2$  is continuous at  $x = 1$ .

### Solution:

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (3x^2 - x + 2) = \lim_{h \rightarrow 0} \left\{ 3(1+h)^2 - (1+h) + 2 \right\} \text{ [Putting } x = 1 + h \text{ as } x \rightarrow 1, h \rightarrow 0] \\ &= \lim_{h \rightarrow 0} (3h^2 + 5h + 4) = 3.0 + 5.0 + 4 = 4 \end{aligned}$$

$$\begin{aligned} \text{and } \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (3x^2 - x + 2) = \lim_{h \rightarrow 0} \left\{ 3(1-h)^2 - (1-h) + 2 \right\} \text{ [Putting } x = 1 - h, h \rightarrow 0 \text{ as } x \rightarrow 1] \\ &= \lim_{h \rightarrow 0} (3h^2 - 5h + 4) = 3.0 - 5.0 + 4 = 4 \end{aligned}$$

$$\text{Again } f(1) = 3.1^2 - 1 + 2 = 4$$

Thus we find that all the values are equal.

$\therefore f(x)$  is continuous at  $x = 1$  ;

### $\varepsilon - \delta$ Definition :

Again corresponding to definition of limit, we may define the continuity of a function as follows :

The functions  $f(x)$  is continuous at  $x = a$ , if  $f(a)$  exists and for any pre-assigned positive quantity  $\varepsilon$ , however small we can determine a positive quantity  $\delta$ , such that  $|f(x) - f(a)| < \varepsilon$ , for all values of  $x$  satisfying  $|x - a| < \delta$ .

Some Properties :

1. The sum or difference of two continuous functions is a continuous function

$$\text{i.e., } \lim_{x \rightarrow a} \{f(x) \pm \phi(x)\} = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} \phi(x).$$

2. Product of two continuous functions is a continuous function.

3. Ratio of two continuous functions is a continuous function, provided the denominator is not zero.

**Continuity in an Interval, at the End Points:**

A function is said to be continuous over the interval (open or closed) including the end points if it is continuous at every point of the same interval.

Let  $c$  be any point in the interval  $(a, b)$  and if  $\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = f(c)$ , then  $f(x)$  is continuous in the interval  $(a, b)$

A function  $f(x)$  is said to be continuous at the *left end* ' $a$ ' of an interval  $a \leq x \leq b$  if  $\lim_{x \rightarrow 0} f(a+h) = f(a)$ , and at the *right end* ' $b$ ' if  $\lim_{x \rightarrow b^-} f(x) = f(b)$ .

**Discontinuity at a Point:** If at any point  $x = a$  in its domain, at least one of values and  $f(a)$  be different from the others, then  $f(a)$  fails to be continuous at that point, i.e., at  $x = a$ .

**Illustration 48.** Discuss the continuity of  $f(x)$  at  $x = 4$ , where

$$\begin{aligned} f(x) &= 2x + 1, x \neq 4 \\ &= 8, x = 4. \end{aligned}$$

**Solution:**

Here  $\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4} (2x + 1) = 2.4 + 1 = 9$  [ $x \rightarrow 4^+$  means  $x$  is approachig 4 from right i.e  $x > 4$ . Also  $f(x) = 2x + 1$  for  $x \neq 4$ ]

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4} (2x + 1) = 2.4 + 1 = 9$$
 [ $x \rightarrow 4^-$  means  $x$  is approachig 4 from left i.e  $x < 4$ . Also  $f(x) = 2x + 1$  for  $x \neq 4$ ]

$f(4) = 8$ , which is different from the previous two values.

$\therefore f(x)$  is not continuous at  $x = 4$ .

**SOLVED ILLUSTRATIONS**

**Illustration 49.** Show that  $f(x) = 2x + 3$  is continuous at  $x = 1$ .

**Solution:**

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} (2x + 3) = 2.1 + 3 = 5.$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} (2x + 3) = 2.1 + 3 = 5. \text{ and at } x = 1, f(1) = 2.1 + 3 = 5$$

$$\therefore \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1) = 5 ; \quad \therefore f(x) \text{ is continuous at } x = 1.$$

**Illustration 50.** Discuss the continuity of  $f(x) = |x|$  at  $x = 0$ .

**Solution:**

$$\begin{aligned} f(x) = |x| \text{ means } f(x) &= x \text{ for } x > 0 \\ &= 0 \text{ for } x = 0 \\ &= -x \text{ for } x < 0. \end{aligned}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} x = 0 \quad [x \rightarrow 0^+ \text{ means } x \text{ is approaching } 0 \text{ from right i.e. } x > 0 \text{ now. Also } f(x) = x \text{ for } x > 0]$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} (-x) = 0 \quad [x \rightarrow 0^- \text{ means } x \text{ is approaching } 0 \text{ from left i.e. } x < 0 \text{ now. Also } f(x) = -x \text{ for } x < 0]$$

$$f(0) = f(x) \text{ at } x = 0 = 0$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0) = 0 ; \therefore f(x) \text{ is continuous at } x = 0.$$

**Illustration 51.** Show that  $f(x) = \frac{1}{x-1}$  is discontinuous at  $x = 1$ .

**Solution:**

$$f(1) = \frac{1}{1-1} = \frac{1}{0}, \text{ undefined.}$$

$$\text{Now } \lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} \frac{1}{1+h-1} = \lim_{h \rightarrow 0} \frac{1}{h} = +\infty \quad (\text{Putting } x = 1 + h, \text{ As } x \rightarrow 1, h \rightarrow 0)$$

$$\text{And } \lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} \frac{1}{1-h-1} = \lim_{h \rightarrow 0} \frac{1}{-h} = -\infty \quad (\text{Putting } x = 1 - h, \text{ As } x \rightarrow 1, h \rightarrow 0)$$

We find right hand limit is not equal to left hand limit.

$\therefore$  at  $x = 1$ ,  $f(x)$  is discontinuous.

**Illustration 52.** Discuss the continuity at  $x = 2$  where

$$\begin{aligned} f(x) &= 4x + 8, \quad x \neq 2 \\ &= 12, \quad x = 2 \end{aligned}$$

**Solution:**

$$\text{Now } \lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} 4(2+h) + 8 = 16 \quad (\text{Putting } x = 2 + h \text{ As } x \rightarrow 2, h \rightarrow 0)$$

$$\text{And } \lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} 4(2-h) + 8 = 16. \quad [\text{Putting } x = 2 - h, h \rightarrow 0, \text{ As } x \rightarrow 2]$$

Again at  $x = 2$ ,  $f(x) = f(2) = 12$ .

Three values are not equal. Hence  $f(x)$  is discontinuous at  $x = 2$ .

### OBJECTIVE QUESTIONS :

- If  $f(x) = (x-1)(x-2)(x-3)$ , find  $f(0)$  [Ans. - 6]
- If  $f(x) = x + |x|$ , find  $f(-2)$  [Ans. 0]
- If  $f(x) = \frac{x^2 - 6x + 8}{x^2 + 8x + 12}$  find  $f(0)$  [Ans.  $\frac{2}{3}$ ]
- Given  $f(x) = x$ ,  $F(x) = \frac{x^2}{x}$  is  $F(x) = f(x)$  always? [Ans. equal for  $x \neq 0$ ]

5. If  $f(x) = 2 + x$ ,  $x < 3$  [Ans. 4]  
 $= 7 - x$ ,  $x \geq 3$ , find  $f(3)$
6. If  $f(x) = (x - 2)(x - 3)(x + 4)$  find  $f(3)$  [Ans. 0]
7. Given  $f(x) = \sqrt{x}$ , for what value of  $x$ ,  $f(x)$  is unreal? [Ans. Any negative value]
8. Find the range of the function  $f(x) = \frac{x-2}{2-x}$ ,  $x \neq 2$  [Ans. - 1]
9. If  $f(x) = x + 2x^3$ , find  $-f(-x)$  [Ans.  $x + 2x^3$ ]
10.  $f(x) = \frac{|x|}{x}$ ,  $x \neq 0$  and is a real number, find  $|f(c) - f(-c)|$ . [Ans. 2]
11. If  $f(x) = e^{3x+4}$ , find  $f(1) \cdot f(2) \cdot f(5)$  [Ans.  $e^{36}$ ]
12. If  $x = \frac{1}{3}$  find  $f(x) = \frac{1-x}{1+x}$ , [Ans.  $\frac{1}{2}$ ]
13. If  $f(x) = x + |x|$  are  $f(2)$  and  $f(-2)$  equal? [Ans. no]
14. Evaluate : (i)  $\lim_{x \rightarrow 1} \frac{x^2 + 1}{x + 1}$  (ii)  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$  (iii)  $\lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2}$  [Ans. (i) 1, (ii) 2, (iii) does not exist]
15. Evaluate : (i)  $\lim_{x \rightarrow 1} \frac{x^2 + 5x - 6}{x - 1}$  (ii)  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$  [Ans. (i) 7 (ii) 1]
16. Evaluate :  $\lim_{x \rightarrow -1} \frac{x^2 + 4x + 3}{x^2 - 7x - 8}$  [Ans.  $-\frac{2}{9}$ ]
17. Find the value of  $\lim_{x \rightarrow 0} \frac{1 - \sqrt{1-x}}{x}$  [Ans.  $\frac{1}{2}$ ]
18. If  $f(x) = x^2$ , evaluate :  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  [Ans.  $2x$ ]
19. Is the function  $f(x) = |x|$  continuous at  $x = 0$  [Ans. Yes]
20. Evaluate :  $\lim_{x \rightarrow \infty} \frac{6 - 5x^2}{4x + 15x^2}$ . [Ans.  $-\frac{1}{3}$ ]
21. A functions is defined as follows :  
 $f(x) = 2x - 1$ ,  $x < 3$   
 $= k$ ,  $x = 3$   
 $= 8 - x$ ,  $x > 3$ .  
 For what value of  $k$ ,  $f(x)$  is continuous at  $x = 3$ ? [Ans. 5]
22. A function is defined as follows :  
 $f(x) = 1$ ,  $x > 0$   
 $= 0$ ,  $x = 0$   
 $= -1$   $x < 0$ . Is the function is continuous at  $x = 0$  [Ans. no]

## DIFFERENTIATION

### Introduction :

The idea of limit as discussed previously will now be extended at present in differentiating a function  $f(x)$  with respect to  $x$  (the independent variable). For this let us know at first what the term 'increment' means.

**Increment :** By increment of a variable we mean the difference of initial value from the final value.

i.e., Increment = final value – initial value.

Let  $x$  change its value from 1 to 4, increment of  $x = 4 - 1 = 3$ .

Again if  $x$  changes from 1 to  $-2$ , increment =  $-2 - 1 = -3$ .

(i.e., increment may be positive or negative).

**Symbols :** Increment of  $x$  will be denoted by  $h$  or,  $\delta x$  (delta  $x$ ) or  $\Delta x$  (delta  $x$ ) and that of  $y$  will be represented by  $k$  or,  $\delta y$  or,  $\Delta y$ .

If in  $y = f(x)$ , the independent variable  $x$  changes to  $x + \delta x$ , then increment of  $x = x + \delta x - x = \delta x (\neq 0)$ .

So  $y = f(x)$  changes to  $y = f(x + \delta x)$ .

$\therefore$  increment of  $y = f(x + \delta x) - f(x)$  [as  $y = f(x)$ ]

Now the increment ratio  $\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} = \frac{f(x + h) - f(x)}{h}$  [assuming  $\delta x = h$ ]

If the ratio  $\frac{\delta y}{\delta x}$  tends to a limit, as  $\delta x \rightarrow 0$  from either side, then this limit is known as the differential coefficient or derivative of  $y [= f(x)]$  with respect to  $x$ . The operation of finding differential coefficient (or derivative) is called Differentiation.

**Illustration 53.** If  $y = 2x^2$ , then  $y + dy = 2(x + dx)^2$ ,  $\delta y = 2(x + dx)^2 - 2x^2$

$$\therefore \frac{\delta y}{\delta x} = \frac{2(x + \delta x)^2 - 2x^2}{\delta x}$$

$$\text{Again for } y = \frac{1}{x^5}, \delta y = \frac{1}{(x + \delta x)^5} - \frac{1}{x^5}$$

**Definition :** A function  $y = f(x)$  is said to be derivable at  $x$  if  $\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$  or,  $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$  or,  $\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$

exists and equal to  $l$ . Now this limit  $l$  that exists is known as derivative (or differential co-efficient) of  $y [= f(x)]$  with respect to  $x$ .

**Symbols :** Derivative of  $y [= f(x)]$  w.r.t. $x$  (with respect to  $x$ ) is denoted by

$$\frac{dy}{dx} \text{ or, } f'(x), \text{ or, } \frac{d}{dx}[f(x)] \text{ or, } Dy \text{ or, } D[f(x)] \text{ or, } y_1$$

Now  $\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$ , provided this limit exists.

Note :  $\frac{dy}{dx}$  does not mean the product of  $\frac{d}{dx}$  with  $y$ . The notation  $\frac{d}{dx}$  stands as a symbol to denote the operation of differentiation only. Read  $\frac{dy}{dx}$  as 'dee y by dee x'.

### SUMMARY :

The whole process for calculating  $f'(x)$  or  $\frac{dy}{dx}$  may be summed up in the following stages :

1. Let the independent variable  $x$  has an increment  $h$  and then find the new value of the function  $f(x+h)$ .
2. Find  $f(x+h) - f(x)$ .
3. Divide the above value by  $h$  i.e., find  $\frac{f(x+h) - f(x)}{h}$ .
4. Calculate  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$

### SOME USEFUL DERIVATIVES :

1.  $\frac{d}{dx} x^n = nx^{n-1}$
2.  $\frac{d}{dx} \cdot \frac{1}{x^n} = -\frac{n}{x^{n+1}}$
3.  $\frac{d}{dx} e^x = e^x$ . [ $e = \text{constant} = 2.718$  (Approx)]
4.  $\frac{d}{dx} a^x = a^x \log_e a$ .
5.  $\frac{d}{dx} \log_e x = \frac{1}{x}$ .
6.  $\frac{d}{dx} (\log_a x) = \frac{1}{x} \log_a e$ . [ $a = \text{Constant other than 'e'}$ ]
7.  $\frac{dc}{dx} = 0$  ( $c = \text{constant}$ )
8.  $\frac{d}{dx} (u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$  [ $u$  and  $v$  are functions of  $x$ ]
9.  $\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$  [ $u$  and  $v$  are functions of  $x$ ]

This formula is known as Product formula because the function to be differentiated is expressed as product of two functions

$$10. \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \quad [u \text{ and } v \text{ are functions of } x]$$

This formula is known as Division formula because the function to be differentiated is expressed as quotient of two functions.

$$11. \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

**Illustration 54.**  $\frac{d}{dx}(x^4) = 4x^3$ ;  $\frac{d}{dx}(x) = 1 \cdot x^{1-1} = 1x^0 = 1 \cdot 1 = 1$ .

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1 \cdot x^{-1-1} = -1x^{-2} = \frac{-1}{x^2}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2} \cdot x^{-1/2} = \frac{1}{2\sqrt{x}};$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2} \cdot x^{-3/2} = \frac{-1}{2x^{3/2}}; \frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{3/2}) = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x};$$

**Illustration 55.**  $\frac{d}{dx}(x^3 + x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) = 3x^2 + 2x$ .

**Illustration 56.**  $\frac{d}{dx}(x^2 \cdot e^x) = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}x^2 = x^2e^x + e^x \cdot 2x = x^2e^x + 2xe^x$ .

**Illustration 57.**  $\frac{d}{dx}(2x^4) = 2 \frac{d(x^4)}{dx} = 2 \cdot 4x^3 = 8x^3$ .

**Illustration 58.** If  $y = \frac{x^2}{x+1}$  find  $\frac{dy}{dx}$

**Solution:**

Let  $y = \frac{u}{v}$  where  $u = x^2$ ,  $\frac{du}{dx} = \frac{d}{dx}(x^2) = 2x^{2-1} = 2x$

And  $v = (x+1)$ ,  $\frac{dv}{dx} = \frac{d}{dx}(x+1) = \frac{d}{dx}x + \frac{d}{dx}1 = 1 + 0 = 1$

Now  $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2} = \frac{(x+1) \cdot 2x - x^2 \cdot 1}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$

**Illustration 59.** Find  $\frac{dy}{dx}$  of the following functions :

- (i)  $x^4 + 4x$ , (ii)  $3x^5 - 5x^3 + 110$ , (iii)  $-2 + (4/5)x^5 - (7/8)x^8$ .

**Solution:**

(i) Let  $y = x^4 + 4x$ .

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 + 4x) = \frac{d}{dx}(x^4) + \frac{d}{dx}(4x) = 4 \cdot x^{4-1} + 4 \frac{d}{dx}x = 4x^3 + 4$$

(ii) Let  $y = 3x^5 - 5x^3 + 110$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^5 - 5x^3 + 110) = \frac{d}{dx}(3x^5) - \frac{d}{dx}(5x^3) + \frac{d}{dx}110$$

$$= 3 \frac{d}{dx}(x^5) - 5 \frac{d}{dx}(x^3) + 0 \text{ (as 110 is a constant number)}$$

$$= 3.5x^{5-1} - 5.3x^{3-1} = 15x^4 + 15x^2.$$

(iii) Let  $y = -2 + \frac{4}{5}x^5 - \frac{7}{8}x^8$

$$\frac{dy}{dx} = \frac{d}{dx} \left( -2 + \frac{4}{5}x^5 - \frac{7}{8}x^8 \right) = \frac{d}{dx}(-2) + \frac{d}{dx} \left( \frac{4}{5}x^5 \right) - \frac{d}{dx} \left( \frac{7}{8}x^8 \right)$$

$$= 0 + \frac{4}{5} \frac{d}{dx}(x^5) - \frac{7}{8} \frac{d}{dx}(x^8), \text{ (as } -2 \text{ is a constant number)}$$

$$= \frac{4}{5} \cdot 5x^{5-1} - \frac{7}{8} \cdot 8x^{8-1} = 4x^4 - 7x^7.$$

**Illustration 60.** If  $s = ut + \frac{1}{2}ft^2$ , find  $\frac{ds}{dt}$  when  $t = 2$ .

**Solution:**

$$\frac{ds}{dt} = \frac{d}{dt} \left( ut + \frac{1}{2}ft^2 \right) = \frac{d}{dt}(ut) + \frac{d}{dt} \left( \frac{1}{2}ft^2 \right) = u \frac{dt}{dt} + \frac{1}{2}f \frac{d}{dt}(t^2)$$

(here  $u, f, \frac{1}{2}$  are constants &  $t$  is a variable, since we are to differentiate w.r.t.  $t$ )

$$= u \cdot 1 + \frac{1}{2}f \cdot 2t^{2-1} = u + \frac{1}{2} \cdot 2ft = u + ft$$

For  $t = 2$ ,  $\frac{ds}{dt} = u + 2f$ .

### ILLUSTRATIONS RELATED TO PRODUCT FORMULA:

**Illustration 61.** Differentiate  $(x + 1)(2x^3 - 21)$  with respect to  $x$ .

**Solution:**

Let  $y = (x + 1)(2x^3 - 21) = u \cdot v$  where  $u = x + 1$ ,  $v = 2x^3 - 21$

$$\frac{du}{dx} = \frac{d}{dx}(x + 1) = \frac{d(x)}{dx} + \frac{d(1)}{dx} = 1 + 0 = 1$$

$$\frac{dv}{dx} = \frac{d}{dx}(2x^3 - 21) = 2 \frac{d}{dx}(x^3) - \frac{d}{dx}(21) = 2 \cdot 3x^{3-1} - 0 = 6x^2.$$

Now  $\frac{dy}{dx} = \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} = (x + 1) \cdot 6x^2 + (2x^3 - 21) \cdot 1$

$$= 6x^3 + 6x^2 + 2x^3 - 21 = 8x^3 + 6x^2 - 21.$$

**Illustration 62.**  $y = x(x^2 - 1)(x^3 + 2)$ , find  $\frac{dy}{dx}$ .

**Solution:**

Let  $y = uvw$ , where  $u = x$ ,  $v = x^2 - 1$ ,  $w = x^3 + 2$

$$\frac{du}{dx} = 1, \frac{dv}{dx} = 2x \text{ and } \frac{dw}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{d}{dx}(uvw) = vw \frac{du}{dx} + uw \frac{dv}{dx} + uv \frac{dw}{dx}$$

$$= (x^2 - 1)(x^3 + 2) \cdot 1 + x(x^3 + 2) \cdot 2x + x(x^2 - 1) \cdot 3x^2$$

$$= 6x^5 - 4x^3 + 6x^2 \text{ (on simplification).}$$

**Illustration 63.** If  $y = 10^x x^{10}$ , find  $\frac{dy}{dx}$ .

**Solution:**

Let  $y = uv$  where  $u = 10^x$  and  $v = x^{10}$

$$\text{Now } \frac{du}{dx} = \frac{d}{dx}(10^x) = (10^x) \log_e 10 ;$$

$$\text{Again } \frac{dv}{dx} = \frac{d}{dx}(x^{10}) = 10 \cdot x^{10-1} = 10x^9.$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx} = 10^x \cdot 10x^9 + x^{10} \cdot 10^x \log_e 10 = 10^x (10x^9 + x^{10} \log_e 10)$$

#### ILLUSTRATION RELATED TO DIVISION FORMULA:

**Illustration 64.** If  $y = \frac{x-1}{x+1}$ , find  $\frac{dy}{dx}$

**Solution:**

$$\text{Let } y = \frac{u}{v}, \text{ where } u = x-1, \frac{du}{dx} = \frac{d}{dx}(x-1) = \frac{d(x)}{dx} - \frac{d}{dx}(1) = 1-0 = 1$$

$$v = x+1, \frac{dv}{dx} = \frac{d}{dx}(x+1) = \frac{d(x)}{dx} + \frac{d}{dx}(1) = 1+0 = 1$$

$$\therefore \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x+1) \cdot 1 - (x-1) \cdot 1}{(x+1)^2} = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

**Illustration 65.** If  $y = \frac{(x+1)(2x^2-1)}{x^2+1}$ , find  $\frac{dy}{dx}$ .

**Solution:**

Let  $y = \frac{u}{v}$ , where  $u = (x+1)(2x^2-1) = 2x^3 + 2x^2 - x - 1$ .

and  $\frac{du}{dx} = 6x^2 + 4x - 1$ ,

$$v = x^2 + 1; \quad \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(x^2+1)(6x^2+4x-1) - (x+1)(2x^2-1) \cdot 2x}{(x^2+1)^2} = \frac{2x^4 + 3x^3 + 2x - 1}{(x^2+1)^2}$$

### DERIVATIVE OF FUNCTION OF A FUNCTION

A variable  $y$  may be a function of a second variable  $z$ , which again may be a function of a third variable  $x$ .

i.e.,  $y = z^2 + 3$ , and  $z = 2x + 1$

Here  $y$  is a function of  $z$  and  $z$  again a function of  $x$ . Ultimately  $y$  is seen to depend on  $x$ , so  $y$  is called the *function of another function*.

Symbolically, if  $y = f(z)$ ,  $z = \phi(x)$  then  $y = f\{\phi(x)\}$

Theorem. If  $y = f(z)$  and  $z = \phi(x)$  then  $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$  (proof is not shown at present)

**Corr.** If  $u = f(v)$ ,  $v = \phi(w)$ ,  $w = \psi(x)$  then  $\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$

**Illustration 66.** To find  $\frac{dy}{dx}$  for  $y = 2z^2 + 1$ ,  $z = 4x - 2$

**Solution:**

$$\text{Now } \frac{dy}{dz} = 4z \text{ and } \frac{dz}{dx} = 4$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 4z \cdot 4 = 16z = 16(4x - 2) = 64x - 32.$$

**Rule 1.** If  $y = ax + b$ , to find  $\frac{dy}{dx}$ . Let  $y = z$ , and  $z = ax + b$ .

So  $y = f(z)$  and  $z = f(x)$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 1 \cdot (a \cdot 1 + 0) = a$$

**Rule 2.** If  $y = (ax + b)^n$ , to find  $\frac{dy}{dx}$  Let  $y = z^n$  and  $z = ax + b$

Now  $\frac{dy}{dz} = n \cdot z^{n-1}$  and  $\frac{dz}{dx} = a$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = n z^{n-1} \cdot a = na (ax + b)^{n-1}$$

**Illustration 67.** : If  $y = (2x + 5)^4$  find  $\frac{dy}{dx}$

**Solution:**

Let  $y = z^4$ , where  $z = 2x + 5$ .

Now  $\frac{dy}{dz} = 4z^3$ ,  $\frac{dz}{dx} = 2$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 4z^3 \cdot 2 = 4 \cdot 2(2x + 5)^3 = 8(2x + 5)^3$$

**Rule 3.** If  $y = \log u$  ( $u$  is a function of  $x$ ), then to find  $\frac{dy}{dx}$ ;  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

**Illustration 68.**  $y = \log(4x)$ , find  $\frac{dy}{dx}$

**Solution:**

Let  $y = \log u$ , where  $u = 4x$ ,  $\frac{du}{dx} = 4$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du} (\log u) \cdot 4 = \frac{1}{u} \cdot 4 = 4 \cdot \frac{1}{4x} = \frac{1}{x}$$

**Illustration 69.**  $y = \log(1 + \sqrt{x})$ , find  $\frac{dy}{dx}$

**Solution:**

Let  $y = \log u$ , where  $u = 1 + \sqrt{x}$

$$\frac{du}{dx} = 0 + \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}; \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{(1+\sqrt{x}) \cdot 2\sqrt{x}}$$

**Illustration 70.** If  $y = \sqrt{x^2 + 7}$ , find  $\frac{dy}{dx}$ .

**Solution:** Let  $y = \sqrt{z}$ , where  $z = x^2 + 7$ ,  $\frac{dz}{dx} = 2x$ .

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{d}{dz} \cdot \sqrt{z} \cdot 2x = \frac{1}{2\sqrt{z}} \cdot 2x = \frac{x}{\sqrt{z}} = \frac{x}{\sqrt{x^2 + 7}}$$

**Illustration 71.** Given  $y = \sqrt{\frac{2x+1}{x+2}}$ , find  $\frac{dy}{dx}$ .

**Solution:**

$$\text{Let } y = \sqrt{u}, \text{ where } u = \frac{2x+1}{x+2}, \frac{du}{dx} = \frac{(x+2) \frac{d}{dx}(2x+1) - (2x+1) \frac{d}{dx}(x+2)}{(x+2)^2}$$

$$\frac{du}{dx} = \frac{(x+2) \cdot 2 - (2x+1) \cdot 1}{(x+2)^2} = \frac{3}{(x+2)^2}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} = \frac{1}{2} \cdot \sqrt{\frac{x+2}{2x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2} \sqrt{\frac{x+2}{2x+1}} \cdot \frac{3}{(x+2)^2} = \frac{3}{2\sqrt{2x+1} \cdot (x+2)^{3/2}}$$

**Illustration 72.** If  $y = \log \log \log x^2$ , find  $\frac{dy}{dx}$

**Solution:**

Let  $y = \log u$  where  $u = \log v$  and  $v = \log x^2 = 2 \log x$ .

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\log v} = \frac{1}{\log \log x^2}$$

$$\frac{du}{dv} = \frac{1}{v} = \frac{1}{\log x^2} = \frac{1}{2 \log x}$$

$$\frac{dv}{dx} = \frac{2}{x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = \frac{1}{\log \log x^2} \cdot \frac{1}{2 \log x} \cdot \frac{2}{x} = \frac{1}{x \log x \log \log x^2}$$

**Illustration 73.** Differentiate  $x^5$  w.r.t.  $x^2$

**Solution:**

Let  $y = x^5$ ,  $z = x^2$ . So the requirement of the question is  $\frac{d}{d(x^2)}(x^5) = \frac{dy}{dz}$

$$\frac{dy}{dx} = 5x^4, \frac{dz}{dx} = \frac{d}{dx}(x^2) = 2x, \text{ so that } \frac{dx}{dz} = \frac{1}{2x}.$$

$$\text{Now } \frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = 5x^4 \cdot \frac{1}{2x} = \frac{5}{2}x^3.$$

**Illustration 74.**  $y = \log_e (x + \sqrt{x^2 + a^2})$ , find  $\frac{dy}{dx}$ .

**Solution:**

Let  $y = \log u$  where  $u = x + \sqrt{x^2 + a^2}$

$$\frac{dy}{du} = \frac{1}{u} \text{ and } \frac{du}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^2 + a^2)^{1/2} \text{ or, } \frac{du}{dx} = 1 + \frac{1}{2\sqrt{x^2 + a^2}} \cdot 2x$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} \cdot \left( 1 + \frac{x}{\sqrt{x^2 + a^2}} \right) = \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}} = \frac{1}{\sqrt{x^2 + a^2}}.$$

### DERIVATIVE OF IMPLICIT FUNCTION

If  $f(x, y) = 0$  defines  $y$  as a derivable function of  $x$ , then differentiate each term w.r.t.  $x$ . The idea will be clear from the given **Illustration**.

**Illustration 75.** Find  $\frac{dy}{dx}$ , if  $3x^4 - x^2y + 2y^3 = 0$

**Solution:**

Differentiating each term of the functions w.r.t.  $x$  we get,  $3 \cdot 4x^3 - \left( x^2 \frac{dy}{dx} + 2xy \right) + 6y^2 \frac{dy}{dx} = 0$

$$\text{or, } 12x^3 - x^2 \frac{dy}{dx} - 2xy + 6y^2 \frac{dy}{dx} = 0$$

$$\text{or, } (6y^2 - x^2) \frac{dy}{dx} = 2xy - 12x^3 \quad \text{or, } \frac{dy}{dx} = \frac{2xy - 12x^3}{6y^2 - x^2}.$$

### DERIVATIVE OF PARAMETRIC FUNCTION

When each of the variables  $x$  and  $y$  can be expressed in terms of a third variable (say  $t$ ) then the function  $y = f(x)$  is known as parametric function.

For example.  $x = f_1(t)$ ,  $y = f_2(t)$ . Now to find  $\frac{dy}{dx}$  we are to find  $\frac{dy}{dt}$  and  $\frac{dx}{dt}$  so that  $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy/dt}{dx/dt}, \frac{dx}{dt} \neq 0$ .

**Illustration 76.** Find  $\frac{dy}{dx}$  when  $x = 4t$ ,  $y = 2t^2$

**Solution:**

$$\frac{dx}{dt} = 4, \frac{dy}{dt} = 4t, \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t}{4} = t$$

**Illustration 77.** Find  $\frac{dy}{dx}$ , when  $x = \frac{3at}{1+t^3}$ ,  $y = \frac{3at^2}{1+t^3}$

**Solution:**

$$\begin{aligned} \frac{dx}{dt} &= \frac{(1+t^3) \cdot 3a - 3at(3t^2)}{(1+t^3)^2} = \frac{3a(1-2t^3)}{(1+t^3)^2} \\ \frac{dy}{dt} &= \frac{(1+t^3) \cdot 6at - 3at^2(3t^2)}{(1+t^3)^2} = \frac{3at(2-t^3)}{(1+t^3)^2} \\ \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{(1+t^3) \cdot 6at - 3at^2(3t^2)}{(1+t^3)^2} + \frac{3a(1-2t^3)}{(1+t^3)^2} = \frac{3at(2-t^3)}{3a(1+2t^3)} = \frac{t(2-t^3)}{1-2t^3} \end{aligned}$$

## SECOND ORDER DERIVATIVE

**Introduction :**

We have seen that the first order derivative of a function of  $x$ , say  $f(x)$ , may also be a function of  $x$ . This new function of  $x$  also may have a derivative w.r.t.  $x$  which is known as second order derivative of  $f(x)$  i.e. second order derivative is the derivative of first order derivative.

Similarly the derivative of the second order derivative is known as third order derivative and so on up to  $n$ th order.

**Symbols :**

For the function  $y = f(x)$ , the first order derivative w.r.t.  $x$  is denoted by  $\frac{dy}{dx}$  or  $f'(x)$  or  $y_1$  as discussed before.

Now the second order derivative of  $y = f(x)$  is expressed as  $\frac{d^2y}{dx^2}$  or  $f''(x)$  or  $y_2$ . The notation  $\frac{d^2y}{dx^2}$  is read as "dee two  $y$  by dee  $x$  squared".

**Illustration 78.** If  $y = x^4$ , find  $\frac{d^2y}{dx^2}$ .

**Solution:**

$$y = x^4, \frac{dy}{dx} = 4x^3, \text{ again } \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} (4x^3) = 4 \cdot 3x^{3-1} = 12x^2.$$

**Note :** To find  $\frac{d^3y}{dx^3}$ ;  $\frac{d^3y}{dx^3} = \frac{d}{dx} \left( \frac{d^2y}{dx^2} \right) = \frac{d}{dx} (12x^2) = 12 \cdot \frac{d}{dx} (x^2)$

$$= 12 \cdot 2x^{2-1} = 24x \text{ i.e., third order derivative is } 24x.$$

**Illustration 79.** Find  $\frac{d^2y}{dx^2}$  if  $y = \frac{\log x}{x}$ .

**Solution:**

$$\begin{aligned} \frac{dy}{dx} = y_1 &= \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} = \frac{1 - \log x}{x^2}; \\ \frac{d^2y}{dx^2} &= \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{1 - \log x}{x^2} \right) = \frac{x^2 \left( -\frac{1}{x} \right) - (1 - \log x) \cdot 2x}{x^4} \\ &= \frac{-x - 2x(1 - \log x)}{x^4} = \frac{-1 - 2(1 - \log x)}{x^3} = \frac{2 \log x - 3}{x^3}. \end{aligned}$$

**Illustration 80.**

If  $y = x^2 e^x$ , Prove  $\frac{d^2y}{dx^2} = (x^2 + 4x + 2)e^x$

**Solution:**

$$y = x^2 e^x$$

Differentiating both sides w.r.t  $x$

$$\begin{aligned} \frac{dy}{dx} &= x^2 \cdot e^x + e^x \cdot 2x \\ &= (x^2 + 2x)e^x \end{aligned}$$

Again differentiating both sides w.r.t  $x$

$$\begin{aligned} \frac{d^2y}{dx^2} &= (x^2 + 2x) e^x + e^x (2x + 2) \\ &= (x^2 + 2x + 2x + 2)e^x \\ &= (x^2 + 4x + 2)e^x \end{aligned}$$

### FOR IMPLICIT FUNCTION AND PARAMETRIC FORMS :

**Illustration 81.** For  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , find  $\frac{d^2y}{dx^2}$

**Solution:**

$$\text{Diff. Both sides w.r.t. } x \text{ we get } \frac{2x}{a^2} + \frac{2y \cdot y_1}{b^2} = 0 \text{ or } \frac{2x}{a^2} + \frac{2y}{b^2} \cdot y_1 = 0 \text{ or } y_1 = -\frac{b^2}{a^2} \cdot \frac{x}{y}$$

$$\frac{d^2y}{dx^2} = \left( -\frac{b^2}{a^2} \right) \cdot \frac{y \cdot 1 - x \cdot y_1}{y^2} = \left( -\frac{b^2}{a^2} \right) \cdot \frac{y + x \left( \frac{b^2}{a^2} \cdot \frac{x}{y} \right)}{y^2} \quad \text{[putting the value of } y_1]$$

$$= \left( -\frac{b^2}{a^2} \right) \cdot \frac{(a^2 y^2 + b^2 x^2)}{a^2 y^3} = \left( -\frac{b^2}{a^2} \right) \cdot \frac{a^2 b^2}{a^2 y^3} \quad \text{[from the given equation we have } a^2 y^2 + b^2 x^2 = a^2 b^2]$$

$$= -\frac{b^4}{y^3}$$

**Illustration 82.**

If  $y = t^2 + t^3$ ,  $x = t - t^4$ , find  $\frac{d^2y}{dx^2}$ .

**Solution:**

$$\frac{dy}{dt} = 2t + 3t^2, \quad \frac{dx}{dt} = 1 - 4t^3$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{2t + 3t^2}{1 - 4t^3}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) \cdot \frac{dt}{dx} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \bigg/ \frac{dx}{dt} = \frac{d}{dt} \left( \frac{2t + 3t^2}{1 - 4t^3} \right) \bigg/ (1 - 4t^3) \\ &= \frac{(1 - 4t^3)(2 + 6t) - (2t + 3t^2)(-12t^2)}{(1 - 4t^3)^2 \cdot (1 - 4t^3)} = \frac{12t^4 + 16t^3 + 6t + 2}{(1 - 4t^3)^2} \end{aligned}$$

**Illustration 83.** Find gradient to the curve  $y = \sqrt{4 - x^2}$  at the point. Where the ordinate & abscissa are same.

**Solution:**

$$y = \sqrt{4 - x^2}$$

$$\text{Or, } \frac{dy}{dx} = \frac{-2x}{2\sqrt{4 - x^2}} = -\frac{x}{y} = -\frac{x}{x} = -1.$$

**Illustration 84.** If  $f(x) = x^k$ , and  $f(1) = 10$ , find 'k'.

**Solution:**

$$f(x) = Kx^{k-1}$$

$$\text{Or } f(1) = K = 10.$$

**Illustration 85.** If  $f(x) = x^2 - 6x + 8$ , find  $f(5) - f(2)$ .

**Solution:**

$$f(x) = 2x - 6$$

$$f(5) = 10 - 6 = 4$$

$$f(2) = 4 - 6 = -2$$

$$f(5) - f(2) = 4 - (-2) = 6.$$

**Illustration 86.** If  $f(x) = xe^x$ , find x if  $f(x) = 0$

**Solution:**

$$f(x) = x \cdot \frac{d(e^x)}{dx} + e^x \frac{dx}{dx} = 0$$

$$\text{Or, } xe^x + e^x = 0$$

$$\text{Or, } e^x(x+1) = 0 \quad \text{or} \quad x+1 = 0 \quad \text{or} \quad x = -1$$

**Illustration 87.** If  $f(x) = \frac{x}{e^x}$ , find  $x$  if  $f'(x) = 0$ .

**Solution:**

$$f'(x) = \frac{e^x \frac{dx}{dx} - x \frac{d(e^x)}{dx}}{e^{2x}} = 0$$

Or,  $e^x - xe^x = 0$

Or,  $e^x(1 - x) = 0$                       or     $1 - x = 0$                       or     $1 = x$

**Illustration 88.** If  ${}^x c_2 = f(x)$ , find  $f'(1)$ .

**Solution:**

$$f'(x) = \frac{x!}{2!(x-2)!} = \frac{x(x-1)}{2}$$

$$f'(x) = \frac{1}{2} \frac{d(x^2 - x)}{dx} = \frac{1}{2} (2x - 1)$$

$$f'(1) = \frac{1}{2} (2-1) = \frac{1}{2}$$

**Illustration 89.** Find gradient of the curve  $y = \log_e^x$  at  $(1,0)$ .

**Solution:**

$y = \log_e^x$                       Or     $0 = \log_e^x$

Or,  $\log_e^1 = \log_e^x$                       Or,     $1 = x$ .

Or,  $\frac{dy}{dx} = \frac{1}{x} = \frac{1}{1} = 1$ .

**Illustration 90.** If  $xy = 1$ , find  $y^2 + f'(x)$ .

**Solution:**

$$y = \frac{1}{x}$$

Or,  $f'(x) = -\frac{1}{x^2}$

So,  $y^2 + f'(x) = \frac{1}{x^2} - \frac{1}{x^2} = 0$ .

**Illustration 91.** Find gradient of the curve  $y = 3x^2 - 5x$  at  $(1, 2)$ .

**Answer:**

$$\begin{aligned} f'(x) &= 6x - 5 \text{ at } x = 1 \\ &= 6 - 5 = 1. \end{aligned}$$

**Illustration 92.**  $y = 2^{\log^x}$ , find  $f'(x)$ .

**Answer:**

$$y = x.$$

Or,  $f'(x) = \frac{dx}{dx} = 1$ .

# Revenue and Cost Function

## 3.2

### Revenue

The term 'revenue' refers to the receipts obtained by a firm from the sale of certain quantities of a commodity at various prices. The revenue concept relates to total revenue, average revenue and marginal revenue.

- **Total Revenue:** Total revenue refers to the total amount of money that the firm receives from the sale of its products. Thus, total revenue is equal to the quantity sold multiplied by the selling price of the commodity. Algebraically total revenue can be written as:

$$TR = PQ$$

where, TR = Total Revenue; P = Price per unit; Q = Quantity sold

- **Average Revenue:** Average Revenue can be obtained by dividing the total revenue by the number of units sold. Thus, Average Revenue can be written as:

$$\text{Average Revenue} = \frac{\text{Total Revenue}}{\text{Total output sold}}$$

$$\text{Or, } AR = \frac{TR}{Q} = \frac{PQ}{Q} = P$$

Thus, average revenue is the price of the commodity.

- **Marginal Revenue:** Marginal Revenue is the net revenue earned by selling one additional unit of the product produced. In other words, marginal revenue is the addition made to the total revenue by selling one additional unit of the commodity. Putting it in algebraic expression, marginal revenue is the addition to total revenue by selling  $(n + 1)$  units of a product instead of  $n$  units. Therefore,

Marginal Revenue = difference in total revenue in increasing sales from  $(n + 1)$  units to  $n$  Units

$$MR = TR_{n+1} - TR_n$$

In other words Marginal Revenue is the rate of change of Total Revenue with respect to the Quantity sold.

If TR stands for Total Revenue and Q stands for output, the marginal revenue (MR) can be expressed as:

$$MR = \frac{d(TR)}{dQ}$$

Here,  $\frac{d(TR)}{dQ}$  indicates derivative of TR with respect to Q which is also the slope of the total revenue curve. Thus, if total revenue is given to us, we can find out marginal revenue at various levels of output by finding the slopes at the corresponding points on the total revenue curve.

**Illustrations 93.**

A schedule on Total Revenue and Amount of output sold is given. Find out Average Revenue and Marginal Revenue

Quantity of Output sold (units)	Total Revenue (TR)	Average Revenue	Marginal Revenue
0	0		
1	29		
2	54		
3	72		
4	80		
5	80		
6	72		

**Solution:**

Quantity of Output sold (units)	Total Revenue (TR)	Average Revenue $\left(\frac{TR}{Q} = \frac{PQ}{Q} = P\right)$	Marginal Revenue $(MR = TR_{n+1} - TR_n)$
0	0	0	-
1	29	29	29
2	54	27	25
3	72	24	18
4	80	20	8
5	80	16	0
6	72	12	-8

**Illustrations 94.** Consider the following market demand schedule:

Price per unit (₹)	3	5	8	12	15	20
Quantity (unit)	25	21	18	15	10	4

Now determine the AR and TR schedules of the seller

**Solution:**

Quantity of Output sold (Q) (units)	Price per unit (P) (₹)	Total Revenue (TR=P.Q) (₹)	Average Revenue (₹) $\left(\frac{TR}{Q} = \frac{PQ}{Q} = P\right)$
25	3	75	3
21	5	105	5
18	8	144	8
15	12	180	12
10	15	150	15
4	20	80	20

## Cost

The term 'total cost' is the sum of all costs incurred by a firm in producing a certain level of output. In the traditional theory total cost is divided into two groups: **total fixed cost and total variable cost.**

$$TC = TFC + TVC$$

where, TC = Total Cost; TFC = Total Fixed Cost; TVC = Total Variable Cost

- **Total Fixed cost:** There are some cost which do not vary with the level of output produced. These remain fixed even when output changes. These are fixed costs. For eg, rent of land, license fees etc. Suppose, that the land on which a factory is situated has an annual rent of ₹ 12,000. It is obvious that the amount of rent will not have any relationship with the quantity of output produced by the factory. Whether the output is 20 tons or 2,000 tons, the rent will remain fixed at ₹ 12,000. In fact, the rent would be the same, even if the factory is temporarily closed down. For this reason, the rent on land is a fixed cost.
- **Total Variable Cost:** On the other hand, some costs change with the level of output produced. This type of cost rises when output increase, and falls when output decrease. For eg, cost of raw materials, wages and salaries of casual employees, transport cost etc. Suppose in order to produce a huge amount of output, we require higher amount of raw materials. So, the cost of raw materials will increase and hence variable cost will increase.

**Average Cost:** The average cost is obtained by dividing Total cost by the level of output. Thus,

$$\text{Average Cost (AC)} = \frac{\text{Total Cost (TC)}}{\text{Output}}$$

Similarly, 
$$\text{Average Fixed Cost (AFC)} = \frac{\text{Total Fixed Cost (TFC)}}{\text{Output}}$$

$$\text{Average Variable Cost (AVC)} = \frac{\text{Total Variable Cost (TVC)}}{\text{Output}}$$

**Marginal Cost:** The marginal cost is defined as the change in total cost which results from a unit change in output. In other words, marginal cost is the addition to the total cost when the quantity of output produced increases by one unit. In symbols it can be written as:

$$MC = TC_{n+1} - TC_n$$

Mathematically, the marginal cost is the first order derivative of the Total cost function. Denoting total cost by TC and output by Q we have:

$$MC = \frac{d(TC)}{dQ}$$

It is worth noting that marginal cost is independent of fixed cost. Since fixed cost do not change with the level of output, there is no marginal fixed cost when output is increased.

**Profit:** The term 'profit' is defined as the difference between total revenue and total cost. Thus,

$$\text{Profit} = \text{Total Revenue} - \text{Total Cost}$$

In order to obtain the maximum amount of output we differentiate the profit function with respect to the output produced and set the equation equal to zero (First order condition for maximization). Then, we find the second order derivative of the function. If the second order derivative is less than zero then we can conclude that the profit is maximized at that level of output. Thus,

$$\frac{d(\text{Profit})}{dQ} = 0 \quad [\text{First Order Condition for maximisation}]$$

$$\frac{d^2(\text{Profit})}{dQ^2} < 0 \quad [\text{Profit is maximized at } Q_{\max}]$$

$$\text{So for maximum Profit, } \frac{d(\text{Profit})}{dQ} = 0 \Rightarrow \frac{d(\text{TR} - \text{TC})}{dQ} = 0 \Rightarrow \frac{d(\text{TR})}{dQ} - \frac{d(\text{TC})}{dQ} = 0 \Rightarrow \text{MR} - \text{MC} = 0 \Rightarrow \mathbf{MR = MC}$$

Thus profit is maximum at the equilibrium point under perfect competition. In other words maximum profit is obtained when Marginal Cost and Marginal Revenue are equal.

**Break-even Point:** Break-even point is the point at which total cost and total revenue are equal which means that the firm does neither earn any profit nor any loss. The firm faces a no gain no loss situation. Thus, at the *Break-even Point* :

$$\text{Profit} = 0$$

$$\text{Or, Total Cost} = \text{Total Revenue}$$

### Illustrations 95.

For a firm the Total Revenue (TR) and Total Cost (TC) functions are given as TR = 20Q and TC = Q<sup>2</sup> + 4Q + 20 where Q = Output. Find the profit maximizing output and maximum profit.

#### Solution:

$$\begin{aligned} \text{Profit} &= \text{TR} - \text{TC} \\ &= 20Q - Q^2 - 4Q - 20 \\ &= 16Q - Q^2 - 20 \end{aligned}$$

$$\frac{d(\text{Profit})}{dQ} = 16 - 2Q$$

$$\therefore 16 - 2Q = 0 \quad [\text{First Order Condition for maximization}]$$

$$\text{Or, } Q = 8$$

$$\frac{d^2(\text{Profit})}{dQ^2} = \frac{d}{dQ} \left\{ \frac{d(\text{Profit})}{dQ} \right\} = \frac{d}{dQ} (16 - 2Q) = -2 < 0$$

So, the profit is maximized at  $Q = 8$

$$\text{The maximum profit} = 16 \times 8 - 8^2 - 20 = 44$$

**Illustrations 96.** In a firm, price of a product is ₹4 and TC = Q<sup>3</sup> - 15Q<sup>2</sup> + 31Q + 100 where Q = Output. Find the profit maximizing output and maximum profit

#### Solution:

$$\text{Total Revenue} = \text{Price} \times \text{Quantity} = 4Q$$

$$\begin{aligned} \text{Profit} &= \text{TR} - \text{TC} \\ &= 4Q - Q^3 + 15Q^2 - 31Q - 100 \end{aligned}$$

$$= -27Q - Q^3 + 15Q^2 - 100$$

$$\frac{d(\text{Profit})}{dQ} = -27 - 3Q^2 + 30Q$$

$$\therefore 3Q^2 - 30Q + 27 = 0 \quad [\text{First Order Condition for maximization}]$$

$$\text{Or, } Q^2 - 10Q + 9 = 0$$

$$\text{Or, } Q^2 - Q - 9Q + 9 = 0$$

$$\text{Or, } Q(Q-1) - 9(Q-1) = 0$$

$$\text{Or, } (Q-9)(Q-1) = 0$$

$$\text{Or, } Q = 9, 1$$

$$\frac{d^2(\text{Profit})}{dQ^2} = \frac{d}{dQ} \left[ \frac{d(\text{Profit})}{dQ} \right] = \frac{d}{dQ} [-27 - 3Q^2 + 30Q] = -6Q + 30$$

$$\text{At } Q = 1, \text{ Profit} = 24 > 0$$

$$\text{At } Q = 9, \text{ Profit} = -24 < 0$$

So, the profit is maximized at  $Q = 9$

$$\text{The maximum profit at } Q = 9 \text{ is } (-27) \times 9 - 9^3 + 15 \times 9^2 - 100 = 143$$

**Illustrations 97.** A firm has the following functions of Total Revenue (TR) and Total Cost (TC) for Q output TR =  $30Q - Q^2$  and TC =  $(Q^3 - 15Q^2 + 10Q + 100)$ . Find the profit maximizing output and maximum profit and equilibrium price.

**Solution:**

$$\text{Profit} = \text{TR} - \text{TC}$$

$$= 30Q - Q^2 - Q^3 + 15Q^2 - 10Q - 100$$

$$= 20Q - Q^3 + 14Q^2 - 100$$

$$\frac{d(\text{Profit})}{dQ} = 20 - 3Q^2 + 28Q$$

$$\therefore 3Q^2 - 28Q - 20 = 0 \quad [\text{F.O.C for maximization}]$$

$$\text{Or, } Q = \frac{28 \pm \sqrt{28^2 + 4 \times 3 \times 20}}{2 \times 3} \quad [\text{by using Sridhar Acharya Formula}]$$

$$\text{Or, } Q = \frac{28 \pm \sqrt{784 + 240}}{6}$$

$$\text{Or, } Q = \frac{28 \pm 32}{6}$$

$$\text{Or, } Q = 10, -\frac{2}{3}$$

$$\frac{d^2(\text{Profit})}{dQ^2} = \frac{d}{dQ} \left[ \frac{d(\text{Profit})}{dQ} \right] = \frac{d}{dQ} [20 - 3Q^2 + 28Q] = -6Q + 28$$

At  $Q = 10$ , Profit =  $-32 < 0$

At  $Q = -\frac{2}{3}$ , Profit =  $32 > 0$

So, the profit is maximized at  $Q = 10$

The maximum profit at  $Q = 10$  is  $20 \times 10 - 10^3 + 14 \times 10^2 - 100 = 500$

We know that  $P = AR$ , so  $AR = \frac{TR}{Q} = 30 - Q$

As profit is maximized at  $Q = 10$  then  $P = 30 - 10 = ₹ 20$  is the equilibrium price

**Illustrations 98.** A firm has the following functions of Total Revenue (TR) and Total Cost (TC) for Q output  $TR = 26Q - 3Q^2$  and  $TC = 2Q^2 - 4Q + 10$ . Find the profit maximizing output and maximum profit and equilibrium price and Total Revenue at that level of output.

**Solution:**

$$\text{Profit} = TR - TC$$

$$= 26Q - 3Q^2 - 2Q^2 + 4Q - 10$$

$$= 30Q - 5Q^2 - 10$$

$$\frac{d(\text{Profit})}{dQ} = 30 - 10Q$$

$$\therefore 30 - 10Q = 0 \quad [\text{F.O.C for maximization}]$$

$$\text{Or, } Q = 3$$

$$\frac{d^2(\text{Profit})}{dQ^2} = \frac{d}{dQ} \left[ \frac{d(\text{Profit})}{dQ} \right] = \frac{d}{dQ} [30 - 10Q] = -10 < 0$$

So, the profit is maximized at  $Q = 3$

The maximum profit at  $Q = 3$  is  $30 \times 3 - 5 \times 3^2 - 10 = 35$

We know that  $P = AR$ , so  $AR = \frac{TR}{Q} = 26 - 3Q$

As profit is maximized at  $Q = 3$  then  $P = 26 - 3 \times 3 = 17$  Money units. This is the equilibrium price

Total Revenue at this price =  $26 \times 3 - 3 \times 3^2 = 51$  Money units

# Optimization Techniques (Basic Concepts)

## 3.3

### Slope and Curvature

For a differentiable function  $y = f(x)$  the derivative  $\frac{dy}{dx}$  or  $f'(x)$  is often called the first derivative or the first order derivative. Generally this first order derivative  $f'(x)$  is a function of  $x$  and therefore  $f'(x)$  is further differentiable with respect to  $x$  which is called the second order derivative of the function  $y = f(x)$ . This second order derivative is denoted by  $f''(x)$  or  $\frac{d^2y}{dx^2}$ . In the same manner if  $f''(x)$  be differentiable then we can obtain the third order derivative  $f'''(x)$  or  $\frac{d^3y}{dx^3}$ . Thus, proceeding in the similar way one can introduce the  $n^{\text{th}}$  order derivative of  $y = f(x)$  which will be denoted by  $\frac{d^n y}{dx^n}$ .

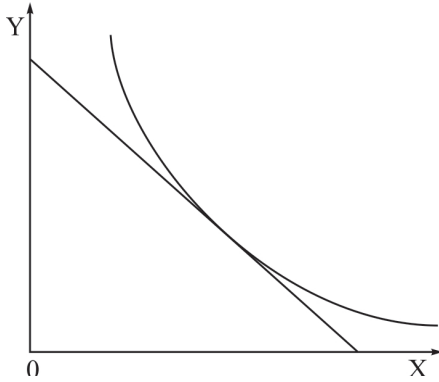
The first and second order derivatives are much used in the graphical analysis. First derivative of a function at any point measures the slope of a function at a particular point. If  $f'(x) > 0$  for all values of  $x$  in the domain of the function then we can say that  $f(x)$  rises continuously from left to right. But we know that the function which rises continuously is called monotonically increasing function. Thus, the condition of a function to be monotonically increasing is that the first derivative must be positive all through. Similarly, a function  $y = f(x)$  is said to be monotonically decreasing if  $f'(x) < 0$  for all  $x$ . In short the condition of monotonicity for a function  $y = f(x)$  is either  $f'(x) > 0$  or  $f'(x) < 0$  for all  $x$ .

### Illustrations 99.

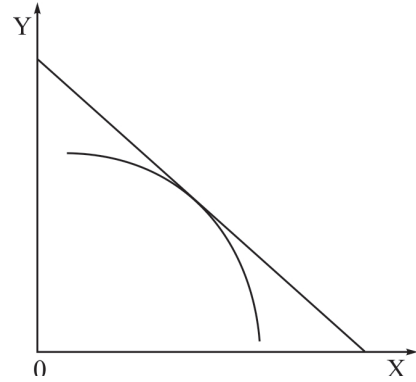
- $f(x) = 2x$   
 $f'(x) = 2 > 0$  This function is monotonically increasing
- $f(x) = -6x$   
 $f'(x) = -6 < 0$ . This function is monotonically decreasing

To know whether a function is increasing at a constant rate or at an increasing rate or at a decreasing rate, we perform the second order derivative. The nature of the function (constant, increasing, decreasing) can be known from the knowledge of the curvature of the curve. By curvature we mean the bending of a curve. To measure this bending we have to measure the rate of change of the slope of the curve. If the rate of change of the slope of the curve is positive i.e.  $f''(x) > 0$  the curve is said to be convex curve. If  $f'(x) > 0$  and  $f''(x) > 0$  the curve increases at an increasing rate. On the other hand, if  $f'(x) < 0$  and  $f''(x) > 0$  the curve decreases at an increasing rate. Lastly, there may be some curve which increase or decreases at a constant rate. Linear curves are examples of this case and for this  $f''(x) = 0$ .

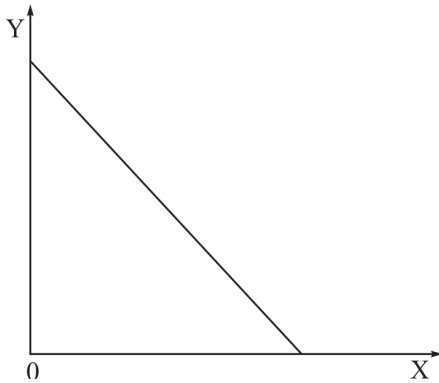
Now, we can also identify whether a curve is convex or concave by just looking at the curve and by drawing a tangent at any point on the curve. If the curve lies above the tangent it will be convex at that point, otherwise if the curve lies entirely below the tangent it will be concave at that point.



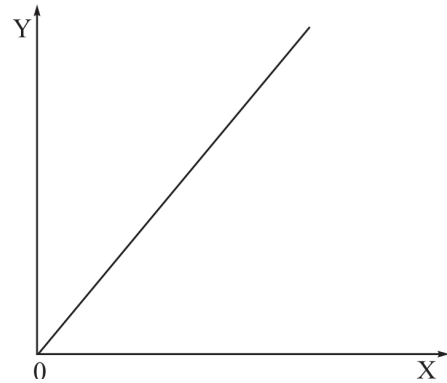
**Fig. 1**  
Convex  
 $f'(x) < 0, f''(x) > 0$



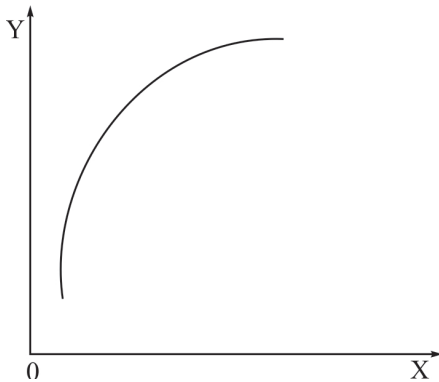
**Fig. 2**  
Concave  
 $f'(x) < 0, f''(x) < 0$



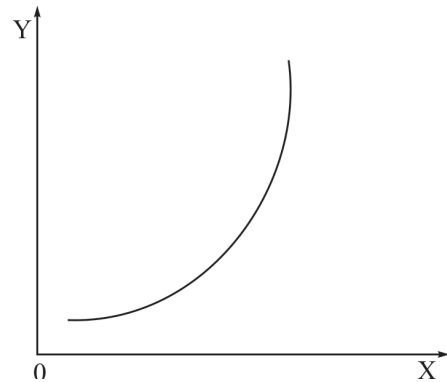
**Fig. 3**  
 $f'(x) < 0, f''(x) = 0$



**Fig. 4**  
 $f'(x) > 0, f''(x) = 0$



**Fig. 5**  
 $f'(x) > 0, f''(x) < 0$



**Fig. 6**  
 $f'(x) > 0, f''(x) > 0$

## Maxima and Minima

In order to determine the relative extremum (relative maximum and minimum) of a function  $y = f(x)$ , the first derivative plays an important role.

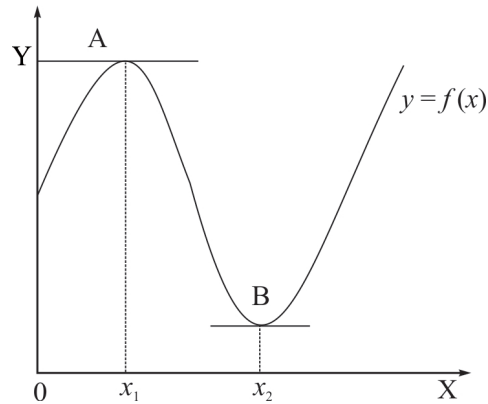


Fig. 7

In the figure 7 given here the function  $y = f(x)$  is attaining the maximum value at the point A, where  $x = x_1$  and minimum value at the point B, where  $x = x_2$ . It appears that the slope of the curve is positive before it attains maximum at the point  $x = x_1$  and is negatively sloped immediately after attaining the maximum value and at the maximum point A the slope of the curve is zero as the tangent at A is horizontal. Since the slope of a function  $y = f(x)$  is represented by its first derivative  $\frac{dy}{dx}$ , so the necessary condition for maximization is  $\frac{dy}{dx} = f'(x) = 0$ . Similarly it appears that the curve is sloping downward before attaining minimum value at B and immediately after attaining the minimum it slopes upward indicating that the slope of the curve is zero at the minimum point B since the tangent at B is horizontal. Since, the slope of the curves at both the extreme point are 0, first order derivative is set at 0 is a necessary condition for both maximization and minimization.

For sufficient condition for maximum and minimum value of the function  $y = f(x)$ , we have to see the slope of the curve immediately after attaining the maximum or minimum. From the figure, it is seen that after attaining maximum point at A the curve is sloping downward while it is sloping upwards after attaining minimum. Thus, a distinction between maximum and minimum can be made by examining the direction of change of the first derivative. Given the function  $y = f(x)$  whose first derivative  $f'(x) = 0$  at point  $x = x_1$  will be a relative maximum if the derivative  $f'(x)$  changes its sign from positive to negative as  $x$  increases in the neighbourhood of  $x = x_1$ . Similarly,  $y = f(x)$  will attain a relative minimum if the derivative  $f'(x)$  changes its sign from negative to positive as  $x$  increases in the neighbourhood of  $x = x_2$ . Since the direction of change of first derivative can be stated in terms of the sign of the second derivative  $f''(x)$ , the sufficient condition for maximization is  $f''(x) < 0$  and for minimization is  $f''(x) > 0$ .

### Illustrations 100.

If  $y = 20x - 2x^2$  then find the value of  $x$  for which  $y$  is optimum.

**Solution:**

$$\frac{dy}{dx} = 20 - 4x .$$

As  $\frac{dy}{dx} = 0$  according to the first order condition,

$$20 - 4x = 0 \text{ or, } 4x = 20 \text{ or, } x = 5$$

$$\frac{d^2y}{dx^2} = -4 < 0 \text{ So, the function is maximum at } x = 5$$

**Illustrations 101.** If  $y = 2x^2 - 16x + 50$ , then find  $x$  where  $y$  is optimum.

**Solution:**

$$\frac{dy}{dx} = 4x - 16.$$

As  $\frac{dy}{dx} = 0$  according to the first order condition,

$$4x - 16 = 0 \quad \text{or, } 4x = 16 \quad \text{or, } x = 4$$

$$\frac{d^2y}{dx^2} = 4 > 0 \quad \text{So, the function is minimum at } x = 4$$

**Illustrations 102.** If  $y = x^3 - 9x^2 + 15x + 20$ , then find the points where  $y$  has maximum and minimum values

**Solution:**

$$\frac{dy}{dx} = 3x^2 - 18x + 15.$$

As  $\frac{dy}{dx} = 0$  according to the first order condition,

$$3x^2 - 18x + 15 = 0$$

$$\text{Or, } 3x^2 - 15x - 3x + 15 = 0$$

$$\text{Or, } 3x(x - 5) - 3(x - 5) = 0$$

$$\text{Or, } (3x - 3)(x - 5) = 0 \quad \text{or, } x = 1, 5$$

$$\frac{d^2y}{dx^2} = 6x - 18$$

Putting  $x = 1$ , we get  $\frac{d^2y}{dx^2} = -12 < 0$

Putting  $x = 5$ , we get  $\frac{d^2y}{dx^2} = 12 > 0$

So, the function is maximum at  $x = 1$  and minimum at  $x = 5$

**Illustrations 103.** Find  $x$  where  $y = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x + 10$  has optimum values.

**Solution:**

$$\frac{dy}{dx} = x^2 - 5x + 4.$$

As  $\frac{dy}{dx} = 0$  according to the first order condition,

$$x^2 - 5x + 4 = 0$$

$$\text{Or, } x^2 - x - 4x + 4 = 0$$

$$\text{Or, } x(x - 1) - 4(x - 1) = 0$$

$$\text{Or, } (x - 4)(x - 1) = 0 \quad \text{or, } x = 1, 4$$

$$\frac{d^2y}{dx^2} = 2x - 5$$

Putting  $x = 1$ , we get  $\frac{d^2y}{dx^2} = -3 < 0$

Putting  $x = 4$ , we get  $\frac{d^2y}{dx^2} = 3 > 0$

So, the function is maximum at  $x = 1$  and minimum at  $x = 4$

## Exercise:

## Theoretical Questions

⊙ Multiple Choice Questions (MCQ)

1.  $\lim_{x \rightarrow 3} (x^3 + 1)$ 
  - (a) 52
  - (b) 53
  - (c) 55
  - (d) 54
  
2.  $\lim_{x \rightarrow 0} (4x^2 + 7x + 5)$ 
  - (a) 4
  - (b) 7
  - (c) 5
  - (d) 16
  
3.  $\lim_{x \rightarrow 4} \left( \frac{x^2 - 16}{x - 4} \right)$ 
  - (a) 2
  - (b) 4
  - (c) 8
  - (d) 6
  
4. If  $y = xe^x$  then  $\frac{dy}{dx} = ?$ 
  - (a)  $xe^x$
  - (b)  $e^x(x+1)$
  - (c)  $e^x(x-1)$
  - (d)  $e^x/x$
  
5.  $f(x) = \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{2} - 7x^2 + 18x$ ,  $f'(x) = ?$ 
  - (a)  $\frac{x^4}{4} + \frac{x^3}{3} + \frac{3x^2}{2} + 14x$
  - (b)  $\frac{x^6}{6} + \frac{x^5}{5} + \frac{x^4}{4} - 7x^3 + 18x$
  - (c)  $x^4 - x^3 - \frac{3x^2}{2} + 14x$
  - (d)  $x^4 + x^3 + \frac{3x^2}{2} - 14x$

6. When  $y = 4^x$  then derivative of  $y$  is ———
- $x(4^{x-1})$
  - $\frac{4^x}{2 \log 2}$
  - $4^x 2 \log 2$
  - None of these
7. Find the differential coefficient of  $y = \frac{e^x}{e^x + 1}$
- $\frac{e^x}{(e^x + 1)^2}$
  - $\frac{e^{2x}}{(e^x + 1)^2}$
  - $\frac{e^x}{(e^{2x} + 1)^2}$
  - $\frac{e^x}{(e^x + 2)^2}$
8.  $y = (4x - 3)^3 + (5x - 2)^2$ . Calculate  $y_1$
- $182x^2 + 13x + 29$
  - $96x^2 + 13x + 29$
  - $12x^2 + 26x + 29$
  - $192x^2 + 26x + 58$
9.  $\frac{x^2}{16} + \frac{y^2}{4} = 1$  is an Implicit function. The derivative of this function is ———
- $\frac{x}{4y}$
  - $-\frac{x}{4y}$
  - $\frac{x}{2y}$
  - $-\frac{x}{2y}$
10. The result of differentiation of  $y = \log[3x^2 + 13x + 10]$  is ———
- $\frac{6x^2 + 13x}{3x^2 + 13x + 10}$
  - $\frac{6x + 13}{3x^3 + 13x^2 + 10x}$
  - $\frac{6x + 13}{3x^2 + 13x + 10}$
  - $\frac{19}{6x + 13}$

11. A soft-drink manufacturer has a revenue function  $R = 7Q^2 - 19Q + 30$  and the cost function is given by  $9Q$ . Find the number of cans produced by the firm, under perfect competition.
- (a) 2  
(b) 4  
(c) 6  
(d) 8
12. A tin manufacturer has a revenue function given by:  $R = 11Q^2 - 110Q + 70$  and the cost function is given by:  $C = 22Q$ . Find the number of tins to be produced by the manufacturer.
- (a) 2  
(b) 6  
(c) 10  
(d) 14
13. A demand function is given by:  $P = a - bQ$  and the cost function is given by  $C = Q^2$ . Find the value of  $Q$  for which profit will be maximum under perfect competition.
- (a)  $\frac{a}{(a+1)}$   
(b)  $\frac{a}{2(b+1)^2}$   
(c)  $\frac{a}{2(b+1)}$   
(d)  $\frac{b}{2(a+1)}$
14. The demand function is given by:  $P = 1400 - 25Q$  and the cost function is given by  $C = 10Q^2$ . Find the value of  $Q$  at the equilibrium point.
- (a) 10  
(b) 20  
(c) 30  
(d) 40
15. A revenue function is given by  $R = 3Q^2 - 8Q + 15$  and the cost function is given by:  $C = 28Q$ . Find the value of  $Q$  for achieving highest profit.
- (a) 3  
(b) 9  
(c) 6  
(d) 12

16. A demand function is given by:  $P = 1500 - 3Q$  and the cost function is given by:  $C = 12Q^2$ . Find the value of  $Q$  at the equilibrium point.
- (a) 50  
 (b) 100  
 (c) 150  
 (d) 125
17. A firm has a fixed production cost of ₹ 90 and a marginal variable production cost of ₹ 9. The price of the product is ₹18. Find the cost function, revenue function, and the value of  $Q$  at the Break Even point.
- (a)  $2Q + 20; 9Q; 10$   
 (b)  $9Q + 90; 18Q; 10$   
 (c)  $4Q + 90; 36Q; 20$   
 (d)  $Q + 10; 5Q; 50$
18. A sugar industry has a fixed cost of ₹290 and a marginal cost of 50 paise. He sells sugar at the price of ₹15/kg. Find the quantity of sugar sold for breaking even.
- (a) 10  
 (b) 30  
 (c) 20  
 (d) 40
19. A cotton mill has a fixed cost of ₹1540 and a marginal cost of ₹33. He sells a shirt at a price of ₹110. Find the minimum number of shirts sold so the mill incurs no loss.
- (a) 23  
 (b) 20  
 (c) 26  
 (d) 29
20. Given:  $C(x) = 900 + 30x + 0.6x^2$ ,  $P = 90$ . Find the value of  $x$  at equilibrium point.
- (a) 40  
 (b) 50  
 (c) 70  
 (d) 30

21. Given:  $C(x) = 2x^2 - 3x - 12$ ,  $P = 33$ . Find the value of  $x$  at equilibrium point.
- (a) 9
  - (b) 12
  - (c) 15
  - (d) 18
22. Given:  $R(x) = 3x^2 + 24x + 2$ ,  $MC(x) = 42$  and the fixed cost is 90. Find the value of  $x$  at equilibrium point.
- (a) 2
  - (b) 3
  - (c) 7
  - (d) 11
23. Given:  $R(x) = 6x^2 - 11x - 35$ ,  $C(x) = 5x^2 - 3x + 16$ . Find the value of  $x$  for which profit is maximum.
- (a) 2
  - (b) 4
  - (c) 6
  - (d) 12
24. Given:  $R(x) = 20x^2 - 15x - 10$ ,  $C(x) = x^2 + 99x + 27$ . Find the value of  $x$  for which profit is maximum.
- (a) 3
  - (b) 15
  - (c) 25
  - (d) 10
25. Given:  $R(x) = 3x^2 + 4x + 2$ ,  $MC(x) = 16$  and the fixed cost is 24. Find the profit maximising value of  $x$  under perfect competition.
- (a) 1
  - (b) 3
  - (c) 4
  - (d) 2
26. A manufacturer has a monthly fixed cost of ₹1,00,000 and a production cost of ₹50 per unit produced. The product is sold at ₹75. Find the cost function and the number of products be sold by the manufacturer to have break even.
- (a)  $25x + 50,000$ ; 2000
  - (b)  $50x + 1,00,000$ ; 4000
  - (c)  $5x + 1,00,000$ ; 3000
  - (d)  $2.5x + 10,000$ ; 5000

27. A cement industry has a yearly fixed cost of ₹96,000 and a monthly production cost of ₹13 per unit produced. The product is sold at ₹39 per unit. Find the cost function
- $13x + 8000$
  - $13x + 96,000$
  - $39x + 96,000$
  - $39x + 8000$
28. Find the monthly profit function if a firm's yearly fixed cost is ₹60,000 and yearly production cost is ₹120 per piece. Each unit is sold at ₹15.
- $\pi = 5x - 5000$
  - $\pi(x) = 15x - 5000$
  - $\pi(x) = 20x - 5000$
  - $\pi(x) = 25x - 5000$
29. Given:  $C(x) = 9x + 350$  and  $P = 14$ . Find the condition of getting break-even point
- $5x - 350 = 0$
  - $7x - 350 = 0$
  - $-14x - 350 = 0$
  - None of these
30. With reference to Q.29 find the break-even quantity
- 50
  - 70
  - 110
  - 100

**From Q.31 to Q.39, find whether the function is maximum or minimum:**

31.  $f(x) = 6x^2 + 11x - 35$
- Maximum
  - Minimum
  - No Curvature
  - None of the above

32.  $f(x) = -4x^2 - 7x - 35$
- (a) Maximum
  - (b) Minimum
  - (c) No Curvature
  - (d) None of the above
33.  $f(x) = 20x^2 - 15x - 10$
- (a) Maximum
  - (b) Minimum
  - (c) No Curvature
  - (d) None of the above
34.  $f(x) = 3x^2 - 4x + 2$
- (a) Maximum
  - (b) Minimum
  - (c) No Curvature
  - (d) None of the above
35.  $f(x) = -x^2 + 6x + 18$
- (a) Maximum
  - (b) Minimum
  - (c) No Curvature
  - (d) None of the above
36.  $f(x) = 9x^2 - 6x + 1$
- (a) Maximum
  - (b) Minimum
  - (c) No Curvature
  - (d) None of the above

37.  $f(x) = -x^2 + 4x - 2$

- (a) Maximum
- (b) Minimum
- (c) No Curvature
- (d) None of the above

38.  $f(x) = 3x^2 + 2$

- (a) Maximum
- (b) Minimum
- (c) No curvature
- (d) None of the above

39.  $f(x) = x^2 - 3x$

- (a) Maximum
- (b) Minimum
- (c) No curvature
- (d) None of the above

**Find the values of  $x$  for which the functions of Q. No. 40 to 50 have maximum and minimum points**

40.  $f(x) = ax^3 + bx^2 + cx + d$ ;  $a < 0$ ;  $b < 0$ ;  $c > 0$  and  $a < b$

(a) Maximum,  $x = \frac{-b + \sqrt{b^2 - 3ac}}{3a}$ ; Minimum,  $x = \frac{-b - \sqrt{b^2 - 3ac}}{3a}$

(b) Minimum,  $x = \frac{-b + \sqrt{b^2 - 3ac}}{3a}$ ; Maximum,  $x = \frac{-b - \sqrt{b^2 - 3ac}}{3a}$

- (c) No Curvature
- (d) None of the above

41.  $f(x) = \frac{x^3}{3} - 9x^2 + 81x + 70$

- (a) Maximum,  $x = 9$ ; Minimum,  $x = 9$
- (b) Minimum,  $x = 3$ ; Maximum,  $x = 5$
- (c) No curvature
- (d) None of the above

$$42. f(x) = \frac{2}{3}x^3 + \frac{9}{2}x^2 - 11x - 21$$

- (a) Maximum,  $x = 2$  ; Minimum,  $x = -\frac{9}{2}$   
 (b) Maximum,  $x = -\frac{11}{2}$  ; Minimum,  $x = 1$   
 (c) Maximum,  $x = -\frac{3}{2}$  ; Minimum,  $x = -3$   
 (d) No curvature

$$43. f(x) = \frac{4}{3}x^3 - 5x^2 + 4x - 9$$

- (a) Maximum,  $x = 4$  ; Minimum,  $x = -\frac{3}{2}$   
 (b) Maximum,  $x = \frac{1}{2}$  ; Minimum,  $x = 2$   
 (c) Maximum,  $x = 2$  ; Minimum,  $x = -1$   
 (d) No curvature

$$44. f(x) = x^3 - 2x^2 - 4x$$

- (a) Maximum,  $x = -\frac{2}{3}$  ; Minimum,  $x = 2$   
 (b) Maximum,  $x = \frac{3}{4}$  ; Minimum,  $x = 1$   
 (c) Maximum,  $x = 3$  ; Minimum,  $x = -\frac{1}{3}$   
 (d) No curvature

$$45. f(x) = \frac{x^3}{3} - 4.5x^2 + 8x + 2$$

- (a) Maximum,  $x = 1$  ; Minimum,  $x = 8$   
 (b) Maximum,  $x = 4$  ; Minimum,  $x = 2$   
 (c) Maximum,  $x = 3$  ; Minimum,  $x = -5$   
 (d) No curvature

$$46. f(x) = \frac{x^3}{3} - \frac{3}{2}x^2 + 2x - 3$$

- (a) Maximum,  $x = 3$  ; Minimum,  $x = 5$   
 (b) Maximum,  $x = 1$  ; Minimum,  $x = 2$   
 (c) Maximum,  $x = 6$  ; Minimum,  $x = 1$   
 (d) No curvature

47.  $f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x$

- (a) Maximum,  $x = -1$  ; Minimum,  $x = \frac{5}{2}$
- (b) Maximum,  $x = 1$  ; Minimum,  $x = 3$
- (c) Maximum,  $x = -1$  ; Minimum,  $x = -\frac{3}{2}$
- (d) No curvature

48.  $f(x) = -\frac{x^3}{3} + 4x^2 - 15x$

- (a) Maximum,  $x = 1$  ; Minimum,  $x = 3$
- (b) Maximum,  $x = 5$  ; Minimum,  $x = 3$
- (c) Maximum,  $x = -3$  ; Minimum,  $x = -5$
- (d) No curvature

49.  $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x$

- (a) Maximum,  $x = -1$  ; Minimum,  $x = 2$
- (b) Maximum,  $x = 1$  ; Minimum,  $x = -2$
- (c) Maximum,  $x = -3$  ; Minimum,  $x = 5$
- (d) No curvature

50.  $f(x) = \frac{x^3}{3} + 2x^2 + 3x + 7$

- (a) Maximum,  $x = -3$  ; Minimum,  $x = -1$
- (b) Maximum,  $x = \frac{1}{2}$  ; Minimum,  $x = 2$
- (c) Maximum,  $x = 1$  ; Minimum,  $x = 4$
- (d) No curvature

**Solution:**

⊙ **Multiple Choice Questions (MCQ)**

1	(c)	Put $x = h + 3$ and then evaluate
2	(c)	Put $x = 0$ and evaluate
3	(c)	Use the formulae $a^2 - b^2 = (a + b)(a - b)$ and then put $x = 4$
4	(b)	By using the formulae
5	(d)	By using the formulae
6	(c)	Use logarithm on both sides
7	(a)	By using the formulae
8	(d)	By using the formulae
9	(b)	By using the formulae
10	(c)	By using the formulae
11	(a)	$MR = \frac{dR}{dQ}$ and $MC = \frac{dC}{dQ}$ , then Put $MR = MC$
12	(b)	$MR = \frac{dR}{dQ}$ and $MC = \frac{dC}{dQ}$ , then Put $MR = MC$
13	(c)	$MR = \frac{dR}{dQ}$ and $MC = \frac{dC}{dQ}$ , then Put $MR = MC$
14	(b)	$MR = \frac{dR}{dQ}$ and $MC = \frac{dC}{dQ}$ , then Put $MR = MC$
15	(c)	$MR = \frac{dR}{dQ}$ and $MC = \frac{dC}{dQ}$ , then Put $MR = MC$
16	(a)	$MR = \frac{dR}{dQ}$ and $MC = \frac{dC}{dQ}$ , then Put $MR = MC$
17	(b)	Profit Function = Revenue Function – Cost Function Put profit function = 0
18	(c)	Profit Function = Revenue Function – Cost Function Put profit function = 0

19	(b)	Profit Function = Revenue Function – Cost Function Put profit function = 0
20	(b)	$MR = \frac{dR}{dx}$ and $MC = \frac{dC}{dx}$ then, Put $MR = MC$
21	(a)	$MR = \frac{dR}{dx}$ and $MC = \frac{dC}{dx}$ then, Put $MR = MC$
22	(b)	$MR = \frac{dR}{dx}$ and $MC = \frac{dC}{dx}$ then, Put $MR = MC$
23	(b)	$MR = \frac{dR}{dx}$ and $MC = \frac{dC}{dx}$ then, Put $MR = MC$
24	(a)	$MR = \frac{dR}{dx}$ and $MC = \frac{dC}{dx}$ then, Put $MR = MC$
25	(d)	$MR = \frac{dR}{dx}$ and $MC = \frac{dC}{dx}$ then, Put $MR = MC$
26	(b)	Profit Function = Revenue Function – Cost Function Put profit function = 0
27	(b)	Total Cost = Fixed Cost + Variable Cost (Production cost)
28	(a)	First divide the production cost by 12 in order to find out the monthly production cost. Then, profit function = Revenue function – cost function
29	(a)	Revenue Function = Price × Quantity. So, $R(x) = P \times x$ Then Profit Function = Revenue Function – Cost Function = 0
30	(b)	Put Profit Function = 0
31	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
32	(a)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum

33	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
34	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
35	(a)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
36	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
37	(a)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
38	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
39	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
40	(a)	Find $f'(x)$ and put $f'(x) = 0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum

41	(c)	<p>Find <math>f'(x)</math> and put <math>f'(x) = 0</math> to find out the roots.</p> <p>Now find <math>f''(x)</math> and put the roots in <math>f''(x)</math> in order to find out the value of <math>f''(x)</math></p> <p>Then, if <math>f''(x) &gt; 0 \rightarrow</math> Minimum</p> <p>if <math>f''(x) &lt; 0 \rightarrow</math> Maximum and if <math>f''(x) = 0 \rightarrow</math> No curvature</p>
42	(b)	<p>Find <math>f'(x)</math> and put <math>f'(x) = 0</math> to find out the roots.</p> <p>Now find <math>f''(x)</math> and put the roots in <math>f''(x)</math> in order to find out the value of <math>f''(x)</math></p> <p>Then, if <math>f''(x) &gt; 0 \rightarrow</math> Minimum</p> <p>if <math>f''(x) &lt; 0 \rightarrow</math> Maximum</p>
43	(b)	<p>Find <math>f'(x)</math> and put <math>f'(x) = 0</math> to find out the roots.</p> <p>Now find <math>f''(x)</math> and put the roots in <math>f''(x)</math> in order to find out the value of <math>f''(x)</math></p> <p>Then, if <math>f''(x) &gt; 0 \rightarrow</math> Minimum</p> <p>if <math>f''(x) &lt; 0 \rightarrow</math> Maximum</p>
44	(a)	<p>Find <math>f'(x)</math> and put <math>f'(x) = 0</math> to find out the roots.</p> <p>Now find <math>f''(x)</math> and put the roots in <math>f''(x)</math> in order to find out the value of <math>f''(x)</math></p> <p>Then, if <math>f''(x) &gt; 0 \rightarrow</math> Minimum</p> <p>if <math>f''(x) &lt; 0 \rightarrow</math> Maximum</p>
45	(a)	<p>Find <math>f'(x)</math> and put <math>f'(x) = 0</math> to find out the roots.</p> <p>Now find <math>f''(x)</math> and put the roots in <math>f''(x)</math> in order to find out the value of <math>f''(x)</math></p> <p>Then, if <math>f''(x) &gt; 0 \rightarrow</math> Minimum</p> <p>if <math>f''(x) &lt; 0 \rightarrow</math> Maximum</p>
46	(b)	<p>Find <math>f'(x)</math> and put <math>f'(x) = 0</math> to find out the roots.</p> <p>Now find <math>f''(x)</math> and put the roots in <math>f''(x)</math> in order to find out the value of <math>f''(x)</math></p> <p>Then, if <math>f''(x) &gt; 0 \rightarrow</math> Minimum</p> <p>if <math>f''(x) &lt; 0 \rightarrow</math> Maximum</p>
47	(a)	<p>Find <math>f'(x)</math> and put <math>f'(x) = 0</math> to find out the roots.</p> <p>Now find <math>f''(x)</math> and put the roots in <math>f''(x)</math> in order to find out the value of <math>f''(x)</math></p> <p>Then, if <math>f''(x) &gt; 0 \rightarrow</math> Minimum</p> <p>if <math>f''(x) &lt; 0 \rightarrow</math> Maximum</p>

48	(b)	Find $f'(x)$ and put $f'(x) = 0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
49	(a)	Find $f'(x)$ and put $f'(x) = 0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
50	(a)	Find $f'(x)$ and put $f'(x) = 0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum If $f''(x) < 0 \rightarrow$ Maximum



**SECTION - B**  
**FUNDAMENTALS OF BUSINESS STATISTICS**



# STATISTICAL REPRESENTATION OF DATA 4

**This Module includes:**

- 4.1 Diagrammatic Representation of Data**
- 4.2 Frequency Distribution**
- 4.3 Graphical representation of Frequency Distribution – Histogram, Frequency Polygon Curve, Ogive, Pie-chart**

# STATISTICAL REPRESENTATION OF DATA

## **Module Learning Objectives:**

After studying this Module, the students will be able to understand –

- ✦ Meaning and definitions of Statistics
- ✦ Statistical Methods
- ✦ Representation of Information through Data
- ✦ Different analysis of Data

The word ‘Statistics’ has been derived from the Latin word ‘Status’ which means a political state. It has also its root either to the Italian word ‘Statista’ or the German word ‘Statistik’ each one of which means a political state. For several decades, the word ‘statistics’ was associated solely with the display of facts and figures pertaining to the economic, demographic and political situations prevailing in a country, usually, collected and brought out by the local governments.

Statistics is a tool in the hands of mankind to translate complex facts into simple and understandable statements of facts.

### **Meaning and definition of Statistics:**

Meaning of statistics: The word Statistics is used in two different senses - Plural and singular. In its plural form, it refers to the numerical data collected in a systematic manner with some definite aim or object in view such as the number of persons suffering from malaria in different colonies of Delhi or number of unemployed girls in different states of India and so on. In Singular form, the word statistics means the science of statistics that deals with the principles, devices or statistical methods of collecting, analyzing and interpreting numerical data.

Thus, ‘statistics’ when used in singular form refers to that branch of knowledge which implies Applied Mathematics. The science of statistics is an old science and it has developed through ages. This science has been defined in different ways by different authors and even the same author has defined it in different ways on different occasions.

It is impossible to enumerate all the definitions given to statistics both as “Numerical Data i.e., Plural Form: and “Statistical Methods, i.e., Singular Form”. However, we have given below some selected definitions of both the forms.

**Definitions of “Statistics in Plural Form or Numerical Data”:** Different authors have given different definitions of statistics. Some of the definitions of statistics describing it quantitatively or in plural form are:

“Statistics are the classified facts representing the conditions of the people in a state especially those facts which can be stated in number or in a table of numbers or in any tabular or classified arrangement.

This definition is narrow as it is confined only to the collection of data regarding the people in a state. But the following definition given by Secrist is modern and convincing. It also brings out the major characteristics of statistical data.

“By Statistics we mean the aggregate of facts affected to a marked extent by multiplicity of causes, numerically expressed, enumerated or estimated according to reasonable standards of accuracy collected in a systematic manner for a pre-determined purpose and placed in relation to each other”

This definition makes it clear that statistics (in plural form or numerical data) should possess the following characteristics.

- I. Statistics means aggregate of facts
- II. Statistics is affected by a large number of causes
- III. Statistics is always numerically expressed
- IV. Statistics should be enumerated or estimated according to reasonable standards of accuracy
- V. Statistics should be collected in a systematic manner
- VI. Statistics should be collected for a pre-determined purpose
- VII. Statistics should be placed in relation to each other.

### **Statistics as Statistical methods or Statistics in Singular Sense:**

We give below the definitions of statistics used in singular sense, i.e., statistics as statistical methods.

Statistical methods provide a set of tools which can be profitably used by different sciences in the manner they deem fit. The term statistics in this context has been defined differently by different authors.

A few definitions are given below:

“Statistics may be called the science of counting”

This definition covers only one aspect, i.e., counting, but the other aspects such as classification, tabulation, etc., have been ignored. As such, the definition is inadequate and incomplete

“Statistics may be defined as the collection, presentation, analysis and interpretation of numerical data”

This definition given by Croxton and Cowden is simple, clear and concise.

According to this definition, there are four stages – collection of data, presentation of data, analysis of data, and interpretation of data. However, one more stage may be added and that is the organization of data. Thus, there are four stages:

1. **Collection and Organization of data:** There are various methods for collecting the data such as census, sampling, primary and secondary data etc.
2. **Presentation of data:** The mass data collected should be presented in a suitable, concise form as the mass data collected is difficult to understand and analyse
3. **Analysis of data:** The mass data collected should be presented in a suitable, concise form for further analysis. Analysis includes condensation, summarisation conclusion, etc., by means of measures of central tendencies, dispersion, skewness, kurtosis, correlation, regression, etc.
4. **Interpretation of data:** The last step is drawing conclusions from the data collected as the figures do not speak for themselves.

Having briefly discussed some of the definitions of the term statistics and having seen their drawbacks we are now in a position to give a simple and complete definition of the ‘Statistics’ in the following words:

Statistics (as used in the sense of data) are numerical statements of facts capable of analysis and interpretation and the science of statistics is a study of the principles and methods used in the collection, presentation, analysis and interpretation of numerical data in any sphere of enquiry.

### Importance and Scope of Statistics:

- I. Statistics and Economics:** According to Prof. Alfred Marshall, “Statistics are the straws out of which I like every other economist, have to make bricks.” The following are some of the fields of economics where statistics is extensively used.
- (a) **Consumption:** Statistical data of consumption enable us to find out the ways in which people in different strata of society spend their incomes.
  - (b) **Production:** The statistics of production describe the total productivity in the country. This enables us to compare ourselves with other countries of the world.
  - (c) **Exchange:** In the field of exchange, an economist studies markets, laws of prices which are determined by the forces of demand and supply, cost of production, monopoly, competition, banking etc. A systematic study of all these can be made only with the help of statistics
  - (d) **Econometrics:** With the help of econometrics, economics has become exact science. Econometrics is the combination of economics, mathematics and statistics.
  - (e) **Public Finance:** Public finance studies the revenue and expenditure activities of a country. Budget, (a statistical document), fiscal policy, deficit financing, etc., are the concepts of economics which are based on statistics.
  - (f) **Input-Output Analysis:** The input-output analysis is based on statistical data which explain the relationship between the input and the output. Sampling, Time series, Index numbers, Probability, Correlation and Regression are some other concepts which are used in economic analysis.

### II. Statistics and Commerce:

Statistical methods are widely applied in the solution of most of the business and trade activities such as production, financial analysis, costing, manpower, planning, business, market research, distribution and forecasting etc. A shrewd businessman always makes a proper and scientific analysis of the past records in order to predict the future course of the business conditions. Index numbers help in predicting the future course of business and economic events. Statistics or statistical methods help the business establishments in analysing the business activities such as:

- (a) **Organization of Business:** Any businessman makes extensive use of statistical data to arrive at the conclusion which guides him in establishing a new firm or business house
- (b) **Production:** The production department of an organisation prepares the forecast regarding the production of the commodity with the help of statistical tools.
- (c) **Scientific Management and Business Forecasting:** Better and efficient control of a business can be achieved by scientific management with the help of statistical data. “The success of businessman lies on the accuracy of forecast made”. The successful businessman is one whose estimates most closely approach accuracy,” said Prof. Boddington.
- (d) **Purchase:** The price statistics of different markets help the businessman in arriving at the correct decision. Raw material is purchased from those markets only where the prices are low.

**III. Statistics and ‘Auditing and Accounting’:** Statistics is widely used in accounting and auditing.

**IV. Statistics and Economic Planning:** According to Prof. Dickinson, “Economic Planning is making of major decisions – what and how much is to be produced, and to whom it is to be allocated – by the conscious

decisions of a determinate authority on the basis of a comprehensive survey of economy as a whole. “The various documents accompanying preceding and following each of the eight Five Year Plans of India are a standing testimony to the fact that statistics is an indispensable tool in economic planning.

- V. **Statistics and Astronomy:** Statistics were first collected by astronomers for the study of the movement of stars and planets. As there are a few things which are common between physical sciences, and statistical methods, astronomers apply statistical methods to go deep in their study. Astronomers generally take a large number of measurements and in most cases there is some difference between several observations. In order to have the best possible measurement they have to make use of the technique of the law of errors in the form of method of least squares.
- VI. **Statistics and Meteorology:** Statistics is related to meteorology. To compare the present with the past or to forecast for the future either temperature or humidity of air or barometrical pressures etc., it becomes necessary to average these figures and thus to study their trends and fluctuations. All this cannot be done without the use of statistical methods. Thus, the science of statics helps meteorology in a large number of ways.
- VII. **Statistics and Biology:** The development of biological theories has been found to be closely associated with statistical methods. Professor Karl Pearson in his Grammar of Sciences has written, “The whole doctrine of heredity rests on statistical basis”.
- VIII. **Statistics and Mathematics:** Mathematics and Statistics have been closely in touch with each other ever since the 17th Century when the theory of probability was found to have influence on various statistical methods. Bowley was right when he said, “Acknowledge of Statistics is like knowledge of foreign language or of algebra: it may prove of use at any time under any circumstances”.

Thus we observe that:

**“Science without statistics bear no fruit, statistics without sciences have no Root”.**

- IX. **Statistics and Research:** Statistical techniques are indispensable in research work. Most of advancement in knowledge has taken place because of experiments conducted with the help of statistical methods.
- X. **Statistics and natural sciences:** Statistics finds an extensive application in physical sciences, especially in engineering physics, chemistry, geology, mathematics, astronomy, medicine, botany, meteorology, zoology, etc.
- XI. **Statistics and Education:** There is an extensive application of statistics in Education. Statistics is necessary for formulation of policies to start new courses, infrastructure required for new courses consideration of facilities available for new courses etc.
- XII. **Statistics and Business:** Statistics is an indispensable tool in all aspects of business. When a man enters business he enters the profession of forecasting because success in business is always the result of precision in forecasting and failure in business is very often due to wrong expectations which arise in turn due to faulty reasoning and inaccurate analysis of various causes affecting a particular phenomenon. Boddington observes, “The successful businessman is the one, whose estimate most closely approaches the accuracy”.

## LIMITATIONS OF STATISTICS

Statistics and its techniques are widely used in every branch of knowledge. W.I. King rightly says:

“Science of statistics is the most useful servant, but only of great value to those who understand its proper use”. The scope of statistics is very wide and it has great utility; but these are restricted by its limitations. Following are the important limitations of statistics:

1. **Statistics does not deal with individual item:** King says, “Statistics from the very nature of the subject cannot and never will be able to take into account individual cases”. Statistics proves inadequate, where one wants to study individual cases. Thus, it fails to reveal the true position.
2. **Statistics deals with quantitative data:** According to Prof. Horace Secrist, “Some phenomenon cannot be quantitatively measured; honesty, resourcefulness, integrity, goodwill, all important in industry as well as in life, are generally not susceptible to direct statistical measurement”.
3. **Statistical laws are true only on averages.** According to W.I. King, “Statistics largely deals with averages and these may be made up of individual items radically different from each other”.

Statistics are the means and not a solution to a problem.

4. **Statistics does not reveal the entire story:** According to Marshall, “Statistics are the straws, out of which, I, like every other economist, have to make bricks”. Croxton says: “It must not be assumed that statistical method is the only method or use in research; neither should this method be considered the best way to attack for every problem”.
5. **Statistics is liable to be misused:** According to Bowley, “Statistics only furnishes a tool though imperfect, which is dangerous in the hands of those who do not know its use and deficiencies”.

W.I. King states, “Statistics are like clay of which you can make a God or Devil as you please”. He remarks, “Science of Statistics is the useful servant, but only of great value to those who understand its proper use”.

6. **Statically data should be uniform and homogeneous**

### STATISTICAL TOOLS USED IN ECONOMIC ANALYSIS

The following are some of the important statistical techniques which are applied in economic analysis:

- (a) Collection of data
- (b) Tabulation
- (c) Measures of Central Tendency
- (d) Measures of Dispersion
- (e) Time Series
- (f) Probability
- (g) Index Numbers
- (h) Sampling and its uses
- (i) Business Forecasting
- (j) Tests of Significance and analysis of variance
- (k) Statistical Quality Control

#### Collection of Data:

Data that is the information collected through censuses and surveys or in a routine manner or other sources is called a raw data. The word data means information (its literary meaning is given as facts). The adjective raw attached to data indicates that the information thus collected and recorded cannot be put to any use immediately and directly.

It has to be converted into more suitable form or processed before it begins to make sense to be utilized gainfully. A raw data is a statistical data in original form before any statistical techniques are used to redefine, process or summarize it.

There are two types of statistical data:

- (i) Primary data
  - (ii) Secondary data
1. **Primary Data:** It is the data collected by a particular person or organization for his own use from the primary source
  2. **Secondary data:** It is the data collected by some other person or organization for their own use but the investigator also gets it for his use.

#### **Methods of Collecting Primary Data:**

The primary data can be collected by the following methods:

1. **Direct personal observation:** In this method, the investigator collects the data personally and, therefore, it gives reliable and correct information.
2. **Indirect oral investigation:** In this method, a third person is contacted who is expected to know the necessary details about the persons for whom the enquiry is meant.
3. **Estimates from the local sources and correspondence.** Here the investigator appoints agents and correspondents to collect the data
4. **Data through questionnaires.** The data can be collected by preparing a questionnaire and getting it filed by the persons concerned.
5. **Investigations through enumerators.** This method is generally employed by the Government for population census, etc.

#### **Methods of Collecting Secondary data:**

The secondary data can be collected from the following sources:

1. Information collected through newspapers and periodicals.
2. Information obtained from the publications of trade associations.
3. Information obtained from the research papers published by University departments or research bureau or UGC.
4. Information obtained from the official publications of the central and the state governments dealing with crop statistics, industrial statistics, trade and transport statistics etc.
5. Information obtained from the official publications of the foreign governments for international organizations.

**Classification of Data:** The process of arranging things in groups or classes according to their common characteristics and affinities is called the classification of data.

“Classification is the process of arranging data into sequences and groups according to their common characteristics or separating them into different but related parts – Secrist.

Thus classification is the process of arranging the available data into various homogenous classes and sub-classes according to some common characteristics or attributes or objectives of investigation.

**Requisites of a Good Classification:**

The main characteristics of a good classification are:

1. It should be exhaustive
2. It should be unambiguous
3. It should be mutually exclusive
4. It should be stable
5. It should be flexible
6. It should have suitability
7. It should be homogeneous
8. It should be a revealing classification
9. It should be reliable
10. It should be adequate

**Advantages of classification of data:**

- (i) It condenses the data and ignores unnecessary details
- (ii) It facilitates comparison of data
- (iii) It helps in studying the relationships between several characteristics
- (iv) It facilitates further statistical treatments

**Types of Classification of Data:**

There are four types of classification of data:

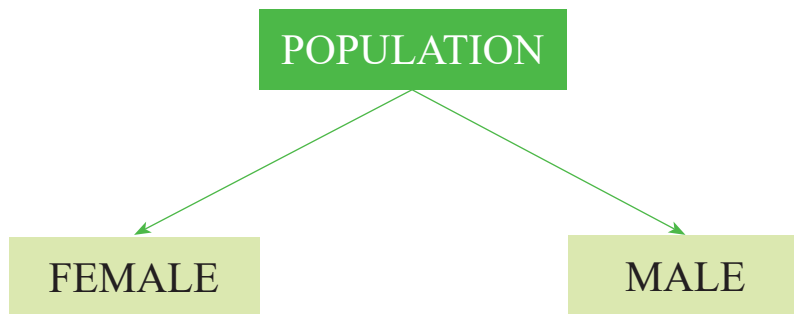
- (i) Quantitative Classification
  - (ii) Temporal Classification
  - (iii) Spatial Classification and
  - (iv) Qualitative Classification
- (i) **Quantitative Classification:** When the basis of classification is according to differences in quantity, the classification is called quantitative
- A quantitative classification refers to classification that is based on figures:** In other words, it is a classification which is based on such characteristics which are quantifiable such as height, weight, marks obtained by students of a class.
- (ii) **Temporal Classification:** When the basis of classification is according to differences in time, the classification is called temporal or chronological classification.
- (iii) **Spatial or Geographical Classification:** When the basis of classification is according to geographic location or place, the classification is called spatial or geographical classification.
- (iv) **Qualitative Classification:** When the basis of classification is according to characteristics or attributes like social status etc. is called qualitative classification.

Classification according to attributes is a method in which the data are divided on the basis of qualities.(i.e., married or single; honest or dishonest; beautiful or ugly; on the basis of religion, viz., Hindu, Muslim, Sikh, Christian etc., known as attributes), which cannot be quantified.

**Qualitative Classification is of two types:**

- (i) Simple Classification or Two-Fold Classification
- (ii) Manifold Classification

1. **Simple Classification or Two-fold Classification:** If the data are classified only into two categories according to the presence or absence of only one attribute, the classification is known as simple or two-fold classification or Dichotomous For example, the population of India may be divided into males and females; literate and illiterate etc.



Moreover, if the classification is done according to a single attribute it is also known as one way classification.

2. **Manifold Classification:** It is a classification where more than one attributes are involved.

**MODE OF PRESENTATION OF DATA**

In this section we shall consider the following three modes of presentation of data

- (a) Textual Presentation
- (b) Tabular Presentation or Tabulation
- (c) Diagrammatic presentation

**Textual Presentation:**

In this method presentation of data is done with the help of a paragraph or a number of paragraphs. The official report of an enquiry commission is usually made by textual presentation. Following are the examples of textual presentation.

**Example1:** In 1995, out of total 2,000 students in a college, 1,400 were for graduation and the rest for post-graduation (P.G.). Out of 1,400 Graduation students 100 were girls. However, in all there were 600 girls in the college. In 2000, number of graduation students increased to 1,700 out of which 250 were girls, but the number of P.G. students fall to 500 of which only 50 were boys. In 2005, out of 800 girls 650 were for graduation, whereas the total number of graduation students was 2,200. The number of boys and girls in P.G. classes were equal.

**Merits and Demerits of Textual Presentation:** The merit of this mode of presentation lies in its simplicity and even a layman can present and understand the data by this method. The observations with exact magnitude can be presented with the help of textual presentation. This type of presentation can be taken as the first step towards the other methods of presentation.

Textual presentation, however, is not preferred by a statistician simply because it is dull, monotonous and comparison between different observations is not possible in this method. For manifold classification, this method cannot be recommended.

**Tabular presentation or tabulation of data:** Tabulation is a scientific process used in setting out the collected data in an understandable form

Tabulation may be defined as logical and systematic arrangement of statistical data in rows and columns. It is designed to simplify the presentation of data for the purposes of analysis and statistical inferences.

Secrist has defined tabulation in the following words:

“Tables are a means of recording in permanent form the analysis that is made through classification and by placing in juxtaposition things that are similar and should be compared”.

The above definition clearly points out that tabulation is a process which gives classification of data in a systematic form and is meant for the purpose of making comparative studies.

Professor Bowley refers to tabulation as:

“The intermediate process between the accumulation of data in whatsoever form they are obtained, and the final reasoned account of the result shown by the statistics”

“Tabulation is the process of condensing classified data in the form of a table so that it may be more easily understood and so that any comparison involved may be more readily made”.

Thus tabulation is one of the most important and ingenious devices of presenting the data in a condensed and readily comprehensible form. It attempts to furnish the maximum information in the minimum possible space, without sacrificing the quality and usefulness of the data.

### **Objectives of Tabulation:**

The purpose of tabulation is to summarise lots of information in such a simple manner that it can be easily analysed and interpreted.

The main objectives of the Tabulation are:

1. To simplify the complex data.
2. To clarify the objective of investigation
3. Economise space.
4. To facilitate comparison
5. To depict trend and pattern of data
6. To act as reference for future studies.
7. To facilitate statistical analysis.
8. To detect errors and omissions in the data
9. To clarify the characteristics of data.

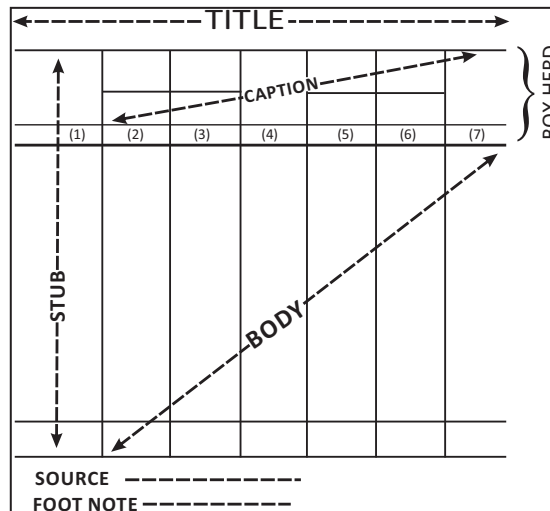
### **Essential Parts of a Statistical Table:**

A good statistical table should invariably have the following parts:

1. **Table Number:** A table should be numbered for identification, especially, when there are a large number of tables in a study. The number may be put at the centre, above the title or at the bottom of the table.

2. **Title of the table:** Every table should have a title. It should be clear, brief and self-explanatory. The title should be set in bold type so as to give prominence.
3. **Date:** The date of preparation of a table should always be written on the table. It enables to recollect the chronological order of the table prepared.
4. **Stubs or Row Designations:** Each row of the table must have a heading. The designations of the rows are called stubs or stub items. Stubs clarify the figures in the rows. As far as possible, the same items should be considered so that they can be included in a single row.
5. **Captions or Column headings:** A table has many columns. Sub-headings of the columns are called captions or headings. They should be well-defined and brief.
6. **Body of the table:** It is the most vital part of the table. It contains the numerical information. It should be made as comprehensive as possible. The actual data should be arranged in such a manner that any figure may be readily located. Different categories of numerical variables should be set out in an ascending order, from left to right in rows and in the same fashion in the columns, from top downwards.
7. **Unit of Measurements:** The unit of measurements should always be stated along with the title, if this is uniform throughout. If different units have been adopted, then they should be stated along the stubs or captions.
8. **Source Notes.** A note at the bottom of the table should always be given to indicate the primary source as well as the secondary source from where the data has been taken, particularly, when there is more than one source.
9. **Foot Notes and References:** It is always placed at the bottom of the table. It is a statement which contains explanation of some specific items, which cannot be understood by the reader from the title, or captions and stubs.

#### Different Parts of Table



**Difference between Textual and Tabular Presentation:** The tabulation method is usually preferred to textual presentation as:

- (i) It facilitates comparison between rows and columns
- (ii) Complicated data can be represented using tabulation
- (iii) Without tabulation, statistical analysis of data is not possible.
- (iv) It is a must for diagrammatic representation.

# Diagrammatic Representation of Data

## 4.1

The representation of statistical data through charts, diagrams and picture is another attractive and alternative method. Unlike the first two methods of representation of data, diagrammatic representation can be used for both the educated section and uneducated section of the society. Furthermore, any hidden trend presented in the given data can be noticed only in this mode of representation.

However, compared to tabulation, this is less accurate. So if there is a priority for accuracy, we have to recommend tabulation.

In this chapter we shall consider the following three types of diagrams:

- I. Line diagram;
- II. Bar diagram;
- III. Pie chart.

### LINE CHART

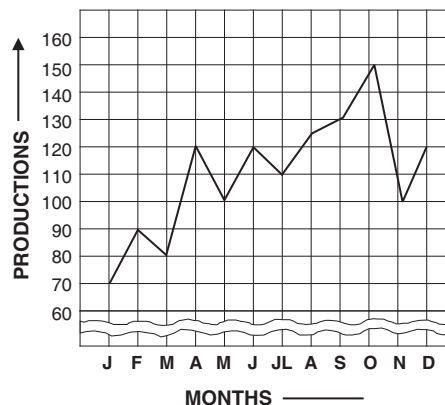
We take a rectangular axes. Along the abscissa, we take the independent variable (x or time) and along the ordinate the dependent variable (y or production related to time). After plotting the points, they are joined by a scale, which represents a line chart. The idea will be clear from the following example.

**Example :** Represent the following data by line chart.

The monthly production of motor cars in India during 2011-12

Jan	Feb	Mar	April	May	June	July	Aug	Sept	Oct	Nov	Dce
70	90	80	120	100	120	110	125	130	150	100	120

**Graph showing production of motor cars.**



## BAR DIAGRAM

The simplest type of graph is the bar diagram. It is especially useful in comparing qualitative data or quantitative data of discrete type. A bar diagram is a graph on which the data are represented in the form of bars. It consists of a number of bars or rectangles which are of uniform width with equal space between them on the x-axis. The height (for vertical bars) or length (for horizontal bars) of the bar is proportional to the value it represents. It should be seen that the bars are neither too short nor too long. The scale should be clearly indicated and baseline be clearly shown.

Bars may be drawn either horizontally or vertically. A good rule to use in determining the direction is that if the legend describing the bar is to be written under the bars then vertical bars should be used; when it is not a requirement, horizontal ones must be used. In this way, the legends can be read without turning the graph. The descriptive legend should not be written at the ends of the bars or within the bars, since such writing may distort the comparison. Usually the diagram will be more attractive if the bars are wider than the space between them.

The width of bars is not governed by any set rules. It is an arbitrary factor. Regarding the space between two bars, it is conventional to have a space about one half of the width of a bar.

The data capable of representation through bar diagrams, may be in the form of row scores, or total scores, or frequencies, or computed statistics and summarised figures like percentages and averages etc.

The bar diagram is generally used for comparison of quantitative data. It is also used in presenting data involving time factor. When two or more sets of data over a certain period of time are to be compared a group bar diagram is prepared by placing the related data side by side in the shape of bars. The bars may be vertical or horizontal in a bar diagram. If the bars are placed horizontally, it is called a Horizontal Bar Diagram. When the bars are placed vertically, it is called a Vertical Bar Diagram.

There are six types of Bar diagram:

- (i) Simple Bar Diagram;
- (ii) Multiple or Grouped Bar Diagram;
- (iii) Subdivided or Component Bar Diagram;
- (iv) Percentage Subdivided Bar Diagram;
- (v) Deviation or Bilateral Bar Diagram;
- (vi) Broken Bars.

### Simple Bar Diagram:

It is used to compare two or more items related to a variable. In this case, the data are presented with the help of bars. These bars are usually arranged according to relative magnitude of bars. The length of a bar is determined by the value or the amount of the variable. A limitation of Simple Bar Diagram is that only one variable can be represented on it.

### Multiple or Grouped Bar Diagram:

A multiple or grouped bar diagram is used when a number of items are to be compared in respect of two, three or more values. In this case, the numerical values of major categories are arranged in ascending or descending order so that the categories can be readily distinguished. Different shades or colours are used for each category.

**Sub-divided or Component Bar Diagram:**

A component bar diagram is one which is formed by dividing a single bar into several component parts.

A single bar represents the aggregate value whereas the component parts represent the component values of the aggregate value. It shows the relationship among the different parts and also between the different parts and the main bar.

**Percentage Sub-divided Bar Diagram:**

It consists of one or more than one bars where each bar totals 100%. Its construction is similar to the subdivided bar diagram with the only difference is that in the sub-divided bar diagram segments are used in absolute quantities and in the percentage bar diagram the quantities are transformed into percentages.

**PIE DIAGRAM OR ANGULAR DIAGRAM**

A pie diagram is a circular graph which represents the total value with its components. The area of a circle represents the total value and the different sectors of the circle represent the different parts. The circle is divided into sectors by radii and the areas of the sectors are proportional to the angles at the centre. It is generally used for comparing the relation between various components of a value and between components and the total value. In pie diagram, the data are expressed as percentages.

Each component is expressed as percentage of the total value. A pie diagram is also known as angular diagram.

The name pie diagram is given to a circle diagram because in determining the circumference of a circle we have to take into consideration a quantity known as 'pie' (written as  $\pi$ ).

**Method of Construction:** The surface area of a circle is known to cover 2 radians or 360 degrees.

The data to be represented through a circle diagram may therefore be presented through 360 degrees or parts or sections of a circle. The total frequencies or value is equated to  $360^\circ$  and then the angles corresponding to component parts are calculated (or the component parts are expressed as percentages of the total and then multiplied by  $360/100$  or 3.6). After determining these angles the required sectors in the circle are drawn. Different shades or colours of designs or different types of hatchings are used to distinguish the various sectors of the circle.

**Illustration 1.**

120 students of a college were asked to opt for different work experiences. The details of these options are as under.

Areas of work experience	No. of students
Photography	6
Clay modeling	30
Kitchen gardening	48
Doll making	12
Book binding	24

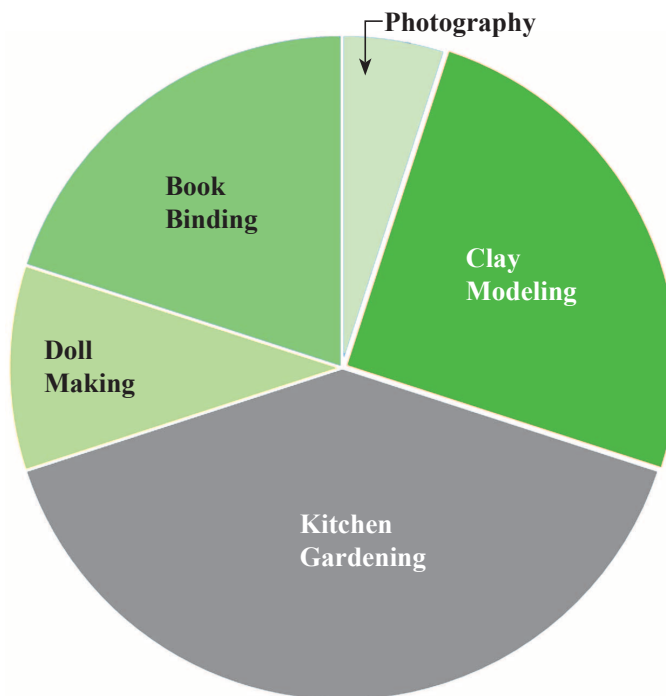
Represent the above data through a pie diagram.

**Solution:**

The numerical data may be converted into the angle of the circles as given below:

Area of Work Experience	No of Students	Angle subtended at the centre of the circle (degrees)
Photography	6	$(6/120) \times 360 = 18$
Clay Modelling	30	$(30/120) \times 360 = 90$
Kitchen Gardening	48	$(48/120) \times 360 = 144$
Doll Making	12	$(12/120) \times 360 = 36$
Book Binding	24	$(24/120) \times 360 = 72$
<b>Total</b>	<b>120</b>	<b>360</b>

With the help of above computations the following Pie diagram is constructed.



# Frequency Distribution

## 4.2

### TALLY BARS AND FREQUENCY

In order to make the data easily understandable, we tabulate the data in the form of tables or charts.

A table has three columns

- (i) Variable
- (ii) Tally marks
- (iii) Frequency

**(i) Variable:** Any character which can vary from one individual to another is called a variable or a variate. For example, age, income, height, intelligence, colour etc. are variates. Some variates are measurable and others are not directly measurable. The examples of measurable variates are age, height, temperature, etc., whereas colour and intelligence are the examples of those variates which cannot be measured numerically. Variables or observations having numerics as possible values are called quantitative variables, whereas those with names of places, quality, and things etc., as possible values are called qualitative variables or attributes.

Variables are of two types i) Continuous; ii) Discontinuous or Discrete. Quantities which can take all numerical values within a certain interval are called continuous variables; But those variables which can take only a finite number of values are called discrete variables; For example, number of students in a particular class, number of sections in a school etc.

**(ii) Tally:** It is a method of keeping count in blocks of five.

For example: | = 1, || = 2, ||| = 3, |||| = 4, ||||| = 5

**Tally Bars:** These are the straight bars used in the Tally.

Each item falling in the class interval, a stroke (vertical Bar) is marked against it. This stroke (Vertical Bar) is called the Tally Bar. Usually, after every four strokes (Tally Bar), in a class, the fifth item is marked by a horizontal or slanted line across the Tally Bars already drawn. For example the frequency 5, 6, 7 is represented by ||||, |||| |, |||| | respectively.

The above method of presentation of data is known as 'Frequency Distribution'. Marks represent the numerical value of the number of students who secured a particular marks and is called Frequency of that particular value of the variate.

In the first column of the table, we write all marks from lowest to highest. We now look at the first mark or value in the given raw data and put a bar (vertical line) in the second column against it. We then, see the second mark or value in the given raw data and put a bar against it in the second column.

This process is repeated till all the observations in the given raw data are exhausted. The bars drawn in the second column are known as tally marks and to facilitate we record tally marks in bunches of five, the fifth tally marks is drawn across the first four.

For example  $\text{|||||} = 8$ . We finally count the number of tally marks corresponding to each observation and write in the third column headed by frequency or number of students.

**(iii) Frequency:** The number of times an observation occurs in the given data is called the frequency of the observation.

**Frequency Distribution:** A frequency distribution is the arrangement of the given data in the form of a table showing frequency with which each variable occurs. In other words, Frequency distribution of a variable is the ordered set  $\{x, f\}$ , where  $f$  is the frequency. It shows all scores in a set of data together with the frequency of each score.

### Types of frequency distributions:

Frequency distributions are of two types:

- (i) Discrete Frequency Distribution
- (ii) Grouped (or Continuous) Frequency Distribution

### Discrete Frequency Distribution

The construction of discrete frequency distribution from the given raw data is done by the method of tally marks as explained earlier.

### Construction of Discrete Frequency Distribution Table

The frequency distribution table has three columns headed by

1. Variables (or classes)
2. Tally Mark or Bars
3. Frequency

The table is constructed by the following steps:

**Step 1:** Prepare three columns, viz., one for the variable (or classes), another for tally marks and the third for the frequency corresponding to the variable (or class).

**Step 2:** Arrange the given data (or values) from the lowest to the highest in the first column under the heading variable (or classes)

**Step 3:** Take the first observation in the raw data and put a bar (or vertical line) in the second column under Tally Marks against it. Then take a second observation and put a tally marks against it, continue this process till all the observations of the given raw data are exhausted. For the sake of convenience, record the tally marks in bunches of five, the fifth bar is placed diagonally crossing the other four (5 is represented by  $\text{||||}$  or  $\text{|||||}$ ) leave some space between each block of bars.

**Step 4:** Count the tally marks of column 2 and place this number opposite to the value of the variable in the third column headed by Frequency.

**Step 5:** Give a suitable title to the frequency distribution table so that it exactly conveys the information contained in the table.

## SOME STATISTICAL TERMS

**Raw Data or Data:** A raw data is a statistical data in original form before any statistical technique is applied to redefine process or summarize it.

**Continuous Variable:** A continuous variable is capable of assuming any value within a certain range or interval. The height, weight, age and temperature of any person can be expressed not only in integral part but also in fractions of any part. For example, the weight of a boy may be 44.0 kg or 44.6 kg or 44.65kg, similarly, his height may be 56 inches or 56.4 inches and age may be 10 years or 10.5 years. Thus, the height, weight, age or temperature etc. are continuous variables.

**Discrete Variable:** A discrete variable can assume only integral values and can have exact measurement. In other words, those variables which can take only a finite set of values are called discrete variables. For example: the number of students in a particular class, or the number of sections in a school, etc. are the examples of discrete variables. Discrete variables are also known as discontinuous variables.

**Continuous Series:** When the continuous variables are arranged in the form of a series, it is called continuous series or exclusive series.

**Discrete or Discontinuous Series:** When the discrete variables are arranged in the form of a series, it is called a discrete or discontinuous series.

**Array:** An array is an arrangement of data in order of magnitude either in descending or ascending order.

**Descending Order:** When data is arranged from the highest value to the lowest value, the array so formed is in descending order.

**Ascending order:** When the data is arranged from the lowest value to the highest value, the array so formed is in ascending order.

### Illustration 2.

If the given data is 17, 7, 11, 5, 13, 9 then

Array in ascending order: 5, 7, 9, 11, 13, 17.

Array in descending order: 17,13,11,9,7, 5.

**Range:** It is the difference between the largest and the smallest number in the given data

The range of the data given in illustration above is  $17 - 5 = 12$ .

**Class, Class-Interval and Class limits.** If the observations of a series are divided into groups and the groups are bounded by limits, then each group is called a class. The end values of a class are called class limits. The smaller value of the two limits is called the lower limit and the higher value of the same is called the upper limit of the class. These two class limits are sometimes called the stated class limits.

**Class Width:** The difference between the lower limit (L) and the upper limit (U) of the class is known as class Width (I).

Thus:  $I = U - L$ .

In other words, the range of a class is called its Class Width.

**Illustration 3.**

The given data is

Marks obtained	Tally Marks	No of Students
1-10		6
11- 20		3
21-30		7
31-40		2
<b>Total</b>		<b>18</b>

In the above data the classes are: 1-10, 11-20, 21-30, and 31-40.

**Class Interval:** The range of the marks from 1 to 40 is grouped into four classes or groups viz: 1 – 10, 11 - 20, 21-30, and 31-40. Each group is known as class interval. The interval between one class and its adjacent class being 9. [As  $10-1 = 9$ ,  $20 - 11 = 9$ ,  $30 - 21 = 9$ , etc.]

**Class Limits:** In the first class 1 – 10, its lower limit is 1 and upper limit is 10. Similarly, 31 is the lower limit and 40 is the upper limit of the class interval 31-40.

**Actual Class Limit or Class Boundaries:** In the illustration, there is a gap of 1 mark between the limits of any two adjacent classes. This gap may be filled up by extending the two limits of each class by half of the value of the gap. Thus

**Lower class boundary = lower class limit –  $\frac{1}{2}$  of the gap**

**Upper class boundary = Upper class limit +  $\frac{1}{2}$  of the gap**

The class boundary of the class 11 to 20 are

**Lower class boundary =  $11 - \frac{1}{2}$  of 1 =  $11 - 0.5 = 10.5$**

**Upper class boundary =  $20 + \frac{1}{2}$  of 1 =  $20 + 0.5 = 20.5$**

In other words, the class boundaries are the limits up to which the two limits, (actual) of each class may be extended to fill up the gap that exists between the classes. The class boundaries of each class, so obtained are called the Actual class limits or True class limits.

True lower class limit = Lower class limit -  $\frac{1}{2}$  of the gap

True upper class limit = Upper class limit +  $\frac{1}{2}$  of the gap

Note: In the case of exclusive series True class limits are the same as class limits

**Illustration 4.**

Class Interval	Class Boundaries
11 – 20	10.5 – 20.5
21 – 30	20.5 – 30.5
31 – 40	30.5 – 40.5

**Class-mark or Mid-point or Mid-value:** The central value of the class interval is called the mid-point or mid-value or class mark. It is the arithmetic mean of the lower class and upper class limit of the same class.

Mid-value of Class = (Lower class limit + Upper class limit)/2

Or,

Class mark = (True Upper class limit + True Lower class limit)/2

The class mark of the class 11-20 is  $(11 + 20)/2 = 15.5$

**Class Magnitude:** It is the difference between the upper class boundary and the lower class boundary of the class. In the illustration the class magnitude of the class, 20.5 – 30.5 is  $(30.5 - 20.5) = 10$ .

**Inclusive and Exclusive Series:** In the above illustration, all the marks we considered were integers.

Hence, it was possible for us to choose classes 11 to 20, 21 to 30 etc. there is a gap of 1 between the upper limit of a class and the lower limit of its succeeding class, which has not created any difficulty. But there can be situations where the raw data is not in integers. For example, in the information regarding maximum temperature of the city or time required to solve a statistical problem is recorded in the data, it may contain fractions as well. In such cases, the consecutive classes have to be necessarily continuous. We have the following:

**Inclusive Series:** When the class-intervals are so fixed that the upper limit of the class is included in that class, it is known as inclusive method of classification, e.g., 0-5, 6-10, 11-15, 16-20.

In the inclusive series, the upper limit and lower limit are included in that class interval. For example, in illustration, the marks 11 and 20 are included in the class 11-20. It is a discontinuous series or inclusive series. In order to make it a continuous one, some adjustment with the class limits is necessary. The class limits are extended to class boundaries by the adjusting adjustment factor, which is equal to half of the difference between the upper limit of the one class and lower limit of the next class. The series so obtained is continuous and is known as exclusive series.

**Exclusive or Continuous Series:** In this series the upper limit of the class is the lower limit of the other class, the common point of the two classes is included in the higher class. For example, 10-15, 15-20, 20-25... represent a continuous series or the exclusive series. In this series, 15 is included in the class 15-20 and 20 is included in 20-30. Here the class intervals overlap and the upper limit of each class is treated as less than that limit and lower limit of each class actually represents exact value. Thus when the class-intervals are so fixed that the upper limit of one class is the lower limit of the next class, it is known as Exclusive method of classification.

### Relative frequency, Percentage frequency and Frequency Density of a class interval

**Relative Frequency:** Frequency of each class can also be expressed as a fraction of total frequency. These are known as relative frequencies. In other words, a relative frequency is the class frequency expressed as a ratio of the total frequency, i.e.

Relative frequency = Class frequency/Total frequency

**Percentage Frequency:** Percentage frequency of a class interval may be defined as the ratio of the class frequency to the total frequency expressed as a percentage.

Percentage frequency = (Class frequency/ Total frequency) × 100

**Frequency Density:** This is defined as the ratio of Frequency of a class to its Width.

Frequency Density = Class Frequency / Class Width

# Graphical Representation of Frequency Distribution

## 4.3

The graphs of frequency distribution are designed to present the characteristic features of a frequency data. They facilitate comparative study of two or more frequency distributions regarding their shape and pattern.

The most commonly used graphs are:

1. Histogram
2. Frequency Polygon
3. Frequency Curve
4. Cumulative Frequency Curve or Ogive.

### Illustration 5.

Draw the pie diagram from the following information

Vehicle Sales by type at a Car Dealers shop are given as:

Type	Numbers Sold
Wagon R	72
S-Cross	54
Amaze	27
Punch	18
City	9
<b>Total</b>	<b>180</b>

### Solution:

The detail computations are as follows:

Type	Nos. Sold	Angle subtended at the centre by the Area (degrees)
Wagon R	72	144
S-Cross	54	108
Amaze	27	54
Punch	18	36
City	9	18
<b>Total</b>	<b>180</b>	<b>360</b>

Angle subtended at the centre of the circle by different Areas (in Degrees) are

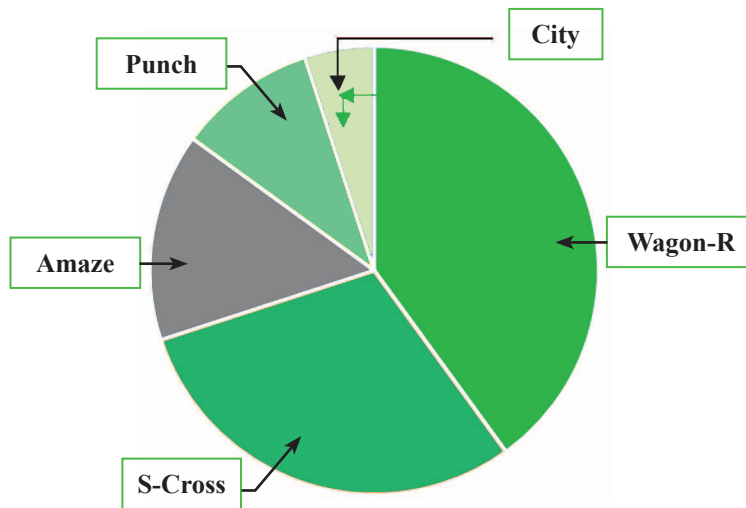
$$\text{Wagon R} = \frac{360}{180} \times 72 = 144$$

$$\text{S-Cross} = \frac{360}{180} \times 54 = 108$$

$$\text{Amaze} = \frac{360}{180} \times 27 = 54$$

$$\text{Punch} = \frac{360}{180} \times 18 = 36$$

$$\text{City} = \frac{360}{180} \times 9 = 18$$



**Illustration 6.**

Prepare a statistical table from the following information of Daily earnings (in ₹) by Doctors:

88	23	27	28	86	96	94	93	86	99
82	24	24	55	88	99	55	86	82	36
96	39	26	54	87	100	56	84	83	46
102	48	27	26	29	100	59	83	84	48
104	46	30	29	40	101	60	89	46	49
106	33	36	30	40	103	70	90	49	50
104	36	37	40	40	106	72	94	50	60
24	39	49	46	66	107	76	96	46	67
26	78	50	44	43	46	79	99	36	68
29	67	56	99	93	48	80	102	32	51

**Solution:**

Detailed calculations are as follows:

Class Interval of Earnings (₹)	Tally Marks	No of Doctors	Class Boundaries
20 – 29		13	19.5 – 29.5
30 – 39		11	29.5 – 39.5
40 – 49		18	39.5 – 49.5
50 – 59		10	49.5 – 59.5
60 – 69		6	59.5 – 60.5
70 – 79		5	60.5 – 79.5
80 – 89		14	79.5 – 89.5
90 – 99		12	89.5 – 99.5
100 – 109		11	99.5 – 109.5
<b>Total</b>	–	<b>100</b>	–

**Illustration 7.**

The national income of a country for the year 1999-2000 to 2001-2002 at current prices was 8650, 9010 and 9530 crores of rupees respectively and per capita income for these years was 250, 256 and 267 rupees. The corresponding figures of national income and per capita income at 1999-2000 prices for the above years were 8650, 8820 and 8850 crores of rupees and 250, 252 and 248 respectively. Present the above data in a table.

**Solution:**

National Income and Per capita income of a country for the year 1999-2000 to 2001-2002

Year	National Income		Per Capita Income	
	At Current Prices (₹ crores)	At 1999-2000 Prices (₹ crores)	At Current Prices	At 1999-2000 Prices
1999-2000	8650	8650	200	250
2000-2001	9010	8820	256	251
2001-2002	9530	8850	267	248

**Illustration 8.**

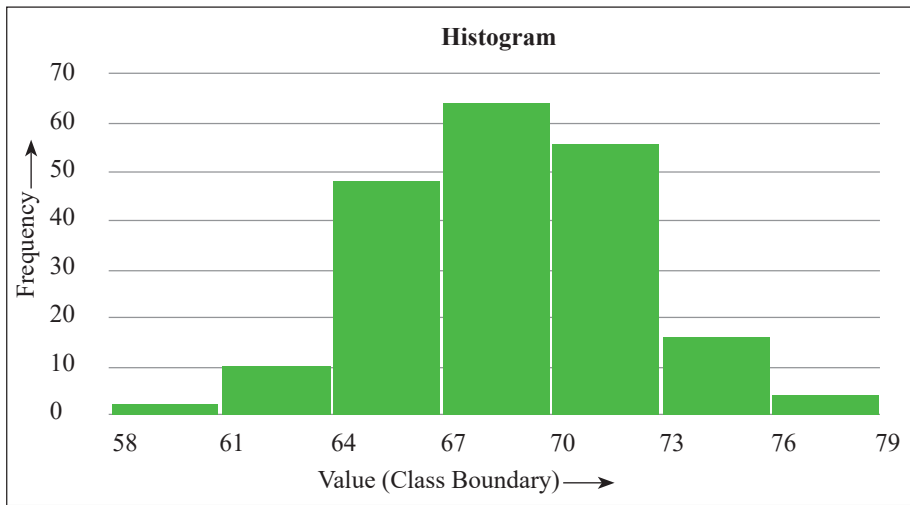
Draw a Histogram of the frequency distribution given below:

Class Interval	Frequency	Class Interval	Frequency
58 – 61	2	70 – 73	56
61 – 64	10	73 – 76	16
64 – 67	48	76 – 79	4
67 – 70	64		

**Solution:**

In this case widths of the classes are of equal magnitude throughout. Width of each class interval is drawn on the X axis i.e. base of each rectangle is proportional to the magnitude of the width of class interval. With width of each class interval as base a rectangle is drawn with height proportional to the corresponding frequency of the class.

Histogram is as follows:



### Illustration 9.

Draw a Histogram of the frequency distribution given below:

Class Interval	Frequency	Class Interval	Frequency
10-14	4	40 – 49	14
15 – 19	12	50 – 74	25
20 – 29	20	75 – 99	10
30 – 39	18		

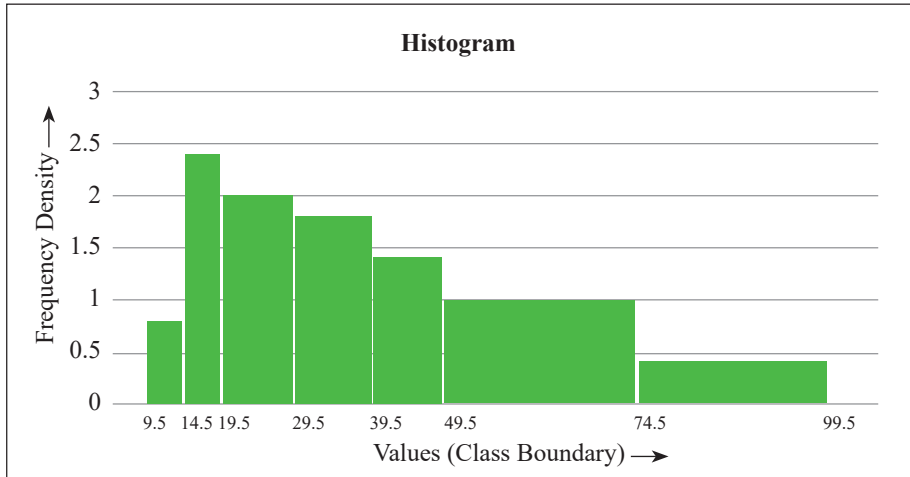
### Solution:

In the given data it is to be noted that the widths of the class intervals are unequal and class boundaries are also not provided. There is a gap of 1 between the limits of any two adjacent classes. Since rectangle in a histogram should have area proportional to the class frequencies, the height of each rectangle should be made proportional to the frequency density. Secondly we can eliminate the gap between the limits of two adjacent classes by bringing in the class boundaries instead of class limits. Thus in this case the class boundaries are 9.5 – 14.5, 14.5 – 19.5 and so on.

### Calculation of Frequency Density

Class Interval Showing Class Limits	Class Interval Showing Class Boundaries	Class Width*	Frequency	Frequency Density
(1)	(2)	(3)	(4)	(5) = (4) ÷ (3)
10 - 14	9.5 - 14.5	5	4	0.8
15 - 19	14.5 - 19.5	5	12	2.4
20 - 29	19.5 - 29.5	10	20	2
30 - 39	29.5 - 39.5	10	18	1.8
40 - 49	39.5 - 49.5	10	14	1.4
50 - 74	49.5 - 74.5	25	25	1
75 - 99	74.5 - 99.5	25	10	0.4

\* Class Width = Upper Class Boundary – Lower Class Boundary



The histogram above was thus drawn by plotting the Values (Class boundary) on X axis and frequency density on Y axis

### Illustration 10.

Draw histogram, frequency polygon and ogives (both “less-than” and “more-than” types) for the following frequency distribution:

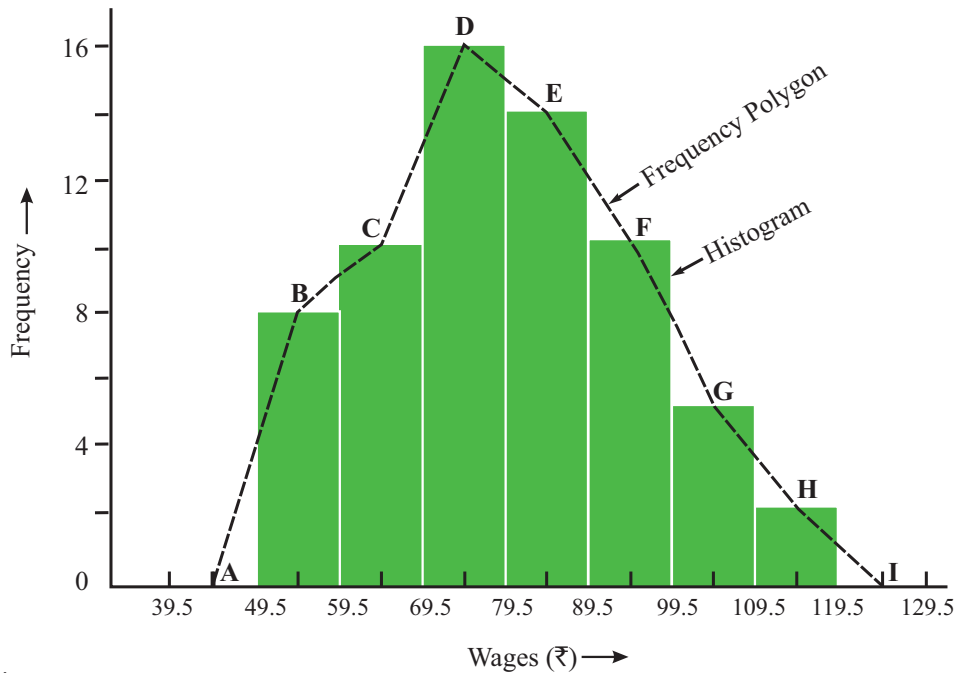
Wages (₹)	50-59	60-69	70-79	80-89	90-99	100-109	110-119
No.of employees	8	10	16	14	10	5	2

### Solution:

[**Note:** Here the class intervals are defined by class limits and so we have to find the class boundaries for drawing the histogram. All the classes have the same width and therefore when drawing the histogram, heights of the rectangles may be represented by the class frequencies. Also for drawing the ogives, we have to calculate both ‘less-than’ and ‘more-than’ cumulative frequencies.]

### Calculations for Drawing Histogram

Class	Class boundaries	Frequency
50 – 59	49.5 – 59.6	8
60 – 69	59.5 – 69.5	10
70 – 79	69.5 – 79.5	16
80 – 89	79.5 – 89.5	14
90 – 99	89.5 – 99.5	10
100 – 109	99.5 – 109.5	5
110 – 119	109.5 – 119.5	2

**Histogram and Frequency Polygon for Wage Distribution**


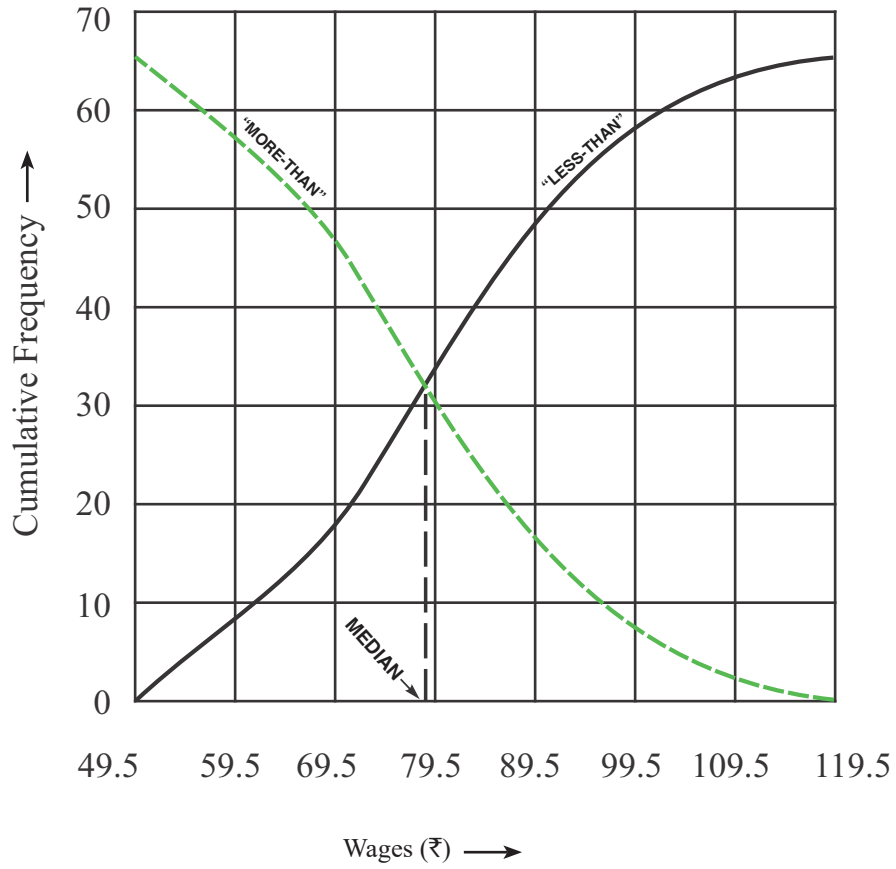
Frequency Polygon is drawn by joining the mid points of the top sides of the frequency bars. Here the points B, C, D, E, F, G and H are the mid points of the frequency bars. Two end points of the Polygon are drawn by considering mid points of two class widths drawn on the two sides of the Histogram. Here A is the mid point of class width considered on the left side of the Histogram and I is that of the width considered on the right side of the Histogram.

Thus for the given data the Frequency Polygon is ABCDEFGHI.

**Calculations for Drawing Ogives**

Class Boundary	Cumulative Frequency	
	'less-than'	'more-than'
49.5	0	65 = $N$
59.5	8	57
69.5	18	47
79.5	34	31
89.5	48	17
99.5	58	7
109.5	63	2
119.5	65 = $N$	0

Ogives for Wage Distribution



**Exercise:****Theoretical Questions**⊙ **Multiple Choice Questions (MCQ)**

1. The word 'Statistics' has been derived from the Latin word 'Status' which means
  - (a) A numerical state
  - (b) A mathematical state
  - (c) A neutral state
  - (d) A political state
  
2. Which one of the following has synonymous words?
  - (a) Status, Staistik, Statista
  - (b) Staistik, Statista, Stats
  - (c) Statistic, Statistia, Stats
  - (d) Statistic, Statistia, Status
  
3. Statistics is a tool in the hands of mankind
  - (a) To translate complex facts into simple and understandable statements of facts
  - (b) To translate statistical information into mathematical statements of facts
  - (c) To translate business information into simple and understandable statements of facts
  - (d) To translate complex facts into presentable forms through tables and diagrams
  
4. The word statistics in plural form means
  - (a) The science of statistics that collect data in a systematic manner with some definite aim or object towards further analysis in graphical way
  - (b) The numerical data collected in a systematic manner with some definite aim or object in view such as the number of persons unemployed in a country.
  - (c) The numerical data collected in a systematic manner for analysing the state of affairs with the help of mathematical applications
  - (d) The science of statistics that deals with the principles, devices or statistical methods of collecting, analyzing and interpreting numerical data

5. The word statistics in singular form means
- (a) The science of statistics that collect data in a systematic manner with some definite aim or object towards further analysis in graphical way
  - (b) The numerical data collected in a systematic manner with some definite aim or object in view such as the number of persons unemployed in a country.
  - (c) The numerical data collected in a systematic manner for analysing the state of affairs with the help of mathematical applications
  - (d) The science of statistics that deals with the principles, devices or statistical methods of collecting, analyzing and interpreting numerical data
6. Which one of the following is not a characteristic of Plural form of Statistics?
- (a) Statistics should be collected for a pre-determined purpose
  - (b) Statistics should be enumerated or estimated
  - (c) Statistics are always graphically expressed
  - (d) Statistics are affected by a large number of causes
8. Measurement of skewness is
- (a) Analysis of data
  - (b) Presentation of data
  - (c) Organisation of data
  - (d) Interpretation of data
9. Find the odd one out from the following
- (a) Regression
  - (b) Kurtosis
  - (c) Sampling
  - (d) Central Tendency
10. Raw data is
- (a) Information which can be interpreted to take decision
  - (b) Information which can't be put to use directly
  - (c) Information which is not amenable to conversion
  - (d) Information which are useless

11. There are four person named A, B, C, & D. A is a sales person whereas B, C, D are students. A collected sales figures for his region and B, C, D used these data in order to study sales pattern. Which one of the following is correct?
- (a) B uses secondary data
  - (b) A & B both are using primary data
  - (c) A, B, C, D all are using secondary data
  - (d) B, C, D are using primary data
12. Which one of the following is a method of collecting primary data?
- (a) Information collected through newspapers and periodicals
  - (b) Information obtained from the publications of trade associations
  - (c) Information collected by Government through Census
  - (d) Information gathered from research paper published in research journal
13. Classification of data is
- (a) The process of arranging things in groups or classes according to their common frequencies
  - (b) The process of arranging things in groups or classes according to their common characteristics and affinities
  - (c) The process of arranging things in groups or classes according to their common differences and tally marks
  - (d) The process of arranging things in groups or classes according to their common deviations from respective mean
14. In Statistics classification
- (a) Separates data into different unrelated parts
  - (b) Separates data into different dispersed groups
  - (c) Separates data into different but related parts
  - (d) Separates data into different modal groups

15. Which one of the following is a characteristic of a good classification?
- (a) Classification should be heterogeneous
  - (b) Classified groups must have overlapping data
  - (c) Classification should be stable
  - (d) Classification should not be inclusive
16. When the basis of classification is according to differences in time it is called
- (a) Temporal Classification
  - (b) Quantitative Classification
  - (c) Spatial Classification
  - (d) Qualitative Classification
17. Dichotomous Classification is
- (a) When data is classified according to presence or absence of two attributes
  - (b) When data is classified into two groups containing all the attributes
  - (c) When data is classified according to presence of two attributes
  - (d) When data is classified into two groups according to presence or absence of one attribute
18. Tabulation Condenses classified data so that
- (a) Data may be more easily understood
  - (b) Data may be easily presented textually
  - (c) Data may be more easily synchronized
  - (d) More comprehensive secondary result could be obtained
19. Tabulation is preferred to textual presentation because
- (a) Tabulation never compares between rows and columns
  - (b) Tabulation is helpful for diagrammatic representation
  - (c) Tabulation always get constructed with simple data
  - (d) None of the above

20. With respect to accuracy

- (a) Diagrammatic presentation is preferable to Tabular presentation
- (b) Textual presentation is preferable to diagrammatic presentation
- (c) Tabular presentation is preferable to Diagrammatic presentation
- (d) Textual presentation is preferable to Tabular presentation

⊙ **State True or False**

1. The class boundaries are the limits up to which the two limits, (actual) of each class may be extended to fill up the gap that exists between the classes;
2. In an exclusive series the class limits are extended to class boundaries by the adjusting adjustment factor;
3. In a Continuous Series lower limit of each class actually represents exact value;
4. Ratio of the class frequency to the total frequency expressed as a percentage is called percentage frequency
5. In Histogram class intervals are taken as heights and corresponding frequencies as breadth;
6.  $\text{Frequency density} = \frac{\text{Class frequency}}{\text{Width of class interval}}$ ;
7. A discrete variable can assume only integral values and therefore is not capable of exact measurement;
8. Variables or observations those with names of places, attributes, and things etc., as possible values are called descriptive variables;
9. A raw data is a statistical data in original form before any statistical technique is applied to redefine process or summarize it;
10. Any character which can vary from one individual to another is called a variable or a variate

**Solution:**

⊙ **Multiple Choice Questions (MCQ)**

1.	2.	3.	4.	5.
(d)	(a)	(a)	(b)	(d)
6.	7.	8.	9.	10.
(c)	(a)	(c)	(b)	(a)
11.	12.	13.	14.	15.
(c)	(b)	(c)	(c)	(a)
16.	17.	18.	19.	
(d)	(a)	(b)	(c)	

⊙ **State True or False**

1	2	3	4	5	6	7	8	9	10
T	T	T	T	F	T	F	F	T	T

# MEASURES OF CENTRAL TENDENCY AND DISPERSION

## 5

### **This Module includes**

- 5.1 Mean, Median, Mode, Mean Deviation**
- 5.2 Range, Quartiles and Quartile Deviation**
- 5.3 Standard Deviation**
- 5.4 Co-efficient of Variation**
- 5.5 Karl Pearson and Bowley's Co-efficient of Skewness**

# MEASURES OF CENTRAL TENDENCY AND DISPERSION

## **Module Learning Objectives:**

After studying this Module, the students will be able to understand the –

- ✦ Concept of Averaging
- ✦ Concept of Arithmetic Mean – different methods of computations with grouped & ungrouped data
- ✦ Concept of Geometric Mean – different methods of computations with grouped & ungrouped data
- ✦ Concept of Harmonic Mean – different methods of computations with grouped & ungrouped data
- ✦ Relation between Arithmetic mean, Geometric mean and Harmonic mean

# Mean, Median, Mode, Mean Deviation

## 5.1

In the previous chapters, data collection and presentation of data were discussed. Even after the data have been classified and tabulated one often finds too much details for many uses that may be made of the information available. We, therefore, frequently need further analysis of the tabulated data. One of the powerful tools of analysis is to calculate a single average value that represents the entire mass of data. The word average is very commonly used in day-to-day conversation. For example, we often talk of average work, average income, average age of employees, etc. An 'Average' thus is a single value which is considered as the most representative or typical value for a given set of data. Such a value is neither the smallest nor the largest value, but is a number whose value is somewhere in the middle of the group. For this reason an average is frequently referred to as a measure of central tendency or central value. Measures of central tendency show the tendency of some central value around which data tends to cluster.

### Objectives of Averaging:

There are two main objectives of the study of averages:

- (i) **To get one single value that describes the characteristic of the entire data.** Measures of central value, can be looked into as condensing the mass of data to one single value which, enable us to get an idea of the entire data. Thus one value can represent thousands, lakhs and even millions of values. For example, it is impossible to remember the individual incomes of millions of earning people of India and even if one could do it there is hardly any use. But if the average income is obtained, we get one single value that represents the entire population. Such a figure would throw light on the standard of living of an average Indian.
- (ii) **To facilitate comparison.** By reducing the mass of data to one single figure, compare various sets of data. Comparison can be made either at a point of time or over a period of time. For example, the figure of average sales for December may be compared with the sales figures of previous months or with the sales figures of another competitive firm.

### Characteristics of a Good Average:

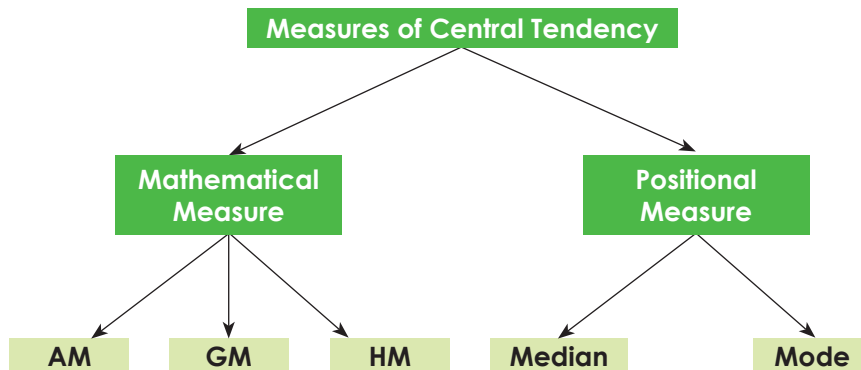
Since an average is a single value representing a group of values, it is desirable that such a value satisfies the following properties:

- (i) **It should be easy to understand.** Since statistical methods are designed to simplify complexity, it is desirable that an average be such that it can be readily understood; otherwise, its use is bound to be very limited.
- (ii) **It should be simple to compute.** Not only an average should be easy to understand but also it should be simple to compute so that it can be used widely. However, though ease of computation is desirable, it should not be sought at the expense of other advantages, i.e., if in the interest of greater accuracy, use of a more difficult average is desirable one should prefer that.

- (iii) **It should be based on all the observations.** The average should depend upon each and every observation so that if any of the observation is dropped average itself is altered.
- (iv) **It should be rigidly defined.** An average should be properly defined so that it has one and only one interpretation. It should preferably be defined by an algebraic formula so that if different people compute the average from the same figures they all get the same answer.
- (v) **It should be capable of further algebraic treatment.** We should prefer to have an average that could be used for further statistical computations. For example, if we are given separately the figures of average income and number of employees of two or more factories we should be able to compute the combined average.
- (vi) **It should have sampling stability.** We should prefer to get a value which has what the statisticians call 'sampling stability'. This means that if we pick 10 different groups of college students, and compute the average of each group, we should expect to get approximately the same values. It does not mean, however, that there can be no difference in the value of different samples. There may be some difference but those averages in which this difference, technically called sampling fluctuation, is less are considered better than those in which this difference is more.
- (vii) **It should not be unduly affected by the presence of extreme values.** If one or two very small or very large observations unduly affect the average, i.e., either increase its value or reduce its value, the average cannot be really considered as typical representative of the entire set of data. In other words, extremes may distort the average and reduce its usefulness.

The following are the important measures of central tendency which are generally used in business:

- A. Arithmetic mean.
- B. Geometric mean,
- C. Harmonic mean
- D. Median
- E. Mode



**I. ARITHMETIC MEAN:** A.M is denoted by  $\bar{X}$ . It is a mathematical measurement. It is calculated by different methods as follows:

(a) **Individual Series/Discrete data:** -

(i) Direct Method: 
$$\bar{X} = \frac{\sum X}{N}$$

(ii) Short Cut Method (or) Indirect Method:  $\bar{X} = A + \frac{\Sigma dx}{N}$

(iii) Step Deviation method:  $\bar{X} = A + \frac{\Sigma dx'}{N} \times i$

Where  $\Sigma X$  is Sum of terms, A is the assumed mean,

$dx$  is the deviation of items from assumed mean A i.e.  $dx = X - A$ ,  $i$  is the common factor,  $dx' = \frac{dx}{i}$ ,  $N$  = No. of observations.

**(b) With grouped data:**

(i) Direct Method:  $\bar{X} = \frac{\Sigma fX}{\Sigma f}$

(ii) Short Cut Method (or) Indirect Method:  $\bar{X} = A + \frac{\Sigma f dx}{\Sigma f}$

(iii) Step Deviation method:  $\bar{X} = A + \frac{\Sigma f dx'}{\Sigma f} \times i$

Where A is the assumed mean,

$dx$  is the deviation of items from assumed mean A

$\Sigma f = N$ ,  $i$  is the common factor in  $dx$ ,  $dx' = \frac{dx}{i}$

**(c) Combined mean:**

When there are two or more than two i.e. multiple series with  $n_1, n_2, n_3, \dots, n_n$  items in respective series with respective means  $\bar{X}_1, \bar{X}_2, \bar{X}_3, \dots, \bar{X}_n$ , then the mean of all the series taken together i.e. combined mean is

$$\bar{X} = \frac{n_1 \bar{X}_1 + n_2 \bar{X}_2 + \dots + n_n \bar{X}_n}{n_1 + n_2 + \dots + n_n}$$

**(d) Weighted mean:**

For calculating simple arithmetic average, we suppose that all the values in the series have equal importance. But in practice this may not be. Different values could have different importance and we assign different weights to different values according to its importance. In such cases we measure weighted mean as per formula:

$$\bar{X} = \frac{w_1 X_1 + w_2 X_2 + \dots + w_n X_n}{w_1 + w_2 + \dots + w_n}$$

Where  $w_1, w_2, w_3, \dots$  are the weights assigned to respective items of a series. Normally frequency values corresponding to the different values of X are taken as the weights.

**Illustration 1.** Find Mean for the following figures:

30, 41, 47, 54, 23, 34, 37, 51, 53, 47

**Solution:**

Sl. No.	1	2	3	4	5	6	7	8	9	10	Total
$x$	30	41	47	54	23	34	37	51	53	47	417

So mean =  $\bar{X} = \frac{\Sigma X}{N} = \frac{417}{10} = 41.7$

**Illustration 2.** Calculate A.M. from the following data:

Marks Obtained	No of Students
4	6
8	12
12	18
16	15
20	9

**Solution:**

**Calculations for A.M**

Marks Obtained (x)	No of Students (f)	fx
4	6	24
8	12	96
12	18	216
16	15	240
20	9	180
Total	60	756

$$\text{So A.M} = \bar{X} = \frac{\sum fX}{\sum f} = \frac{756}{60} = 12.6$$

**Illustration 3.** Use direct method to find  $\bar{X}$

Income (₹)	No of persons
10-20	4
20-30	7
30-40	16
40-50	20
50-60	15
60-70	8

**Solution:**

$$\bar{X} = \frac{\sum fX}{\sum f}$$

Detail computations are as follows:

#### Calculations for $\bar{X}$

Income (₹)	$X$ = mid term of income class interval	No of persons ( $f$ )	$fX$
10-20	15	4	60
20-30	25	7	175
30-40	35	16	560
40-50	45	20	900
50-60	55	15	825
60-70	65	8	520
Total	–	70	3040

$$\bar{X} = \frac{\Sigma fX}{\Sigma f} = \frac{3040}{70} = ₹ 43.43$$

**Illustration 4.**  $\bar{X}$  of 20 terms was found to be 35. But afterwards it was detected that two terms 42 and 34 were misread as 46 and 39 respectively. Find correct  $\bar{X}$ .

**Solution:**

$\bar{X}$  of 20 terms = 35

Incorrect Total of 20 terms =  $35 \times 20 = 700$

Correct Total =  $700 + 42 + 34 - 46 - 39 = 691$

Correct  $\bar{X} = 691/20 = 34.55$

**Illustration 5.** The mean of wages in factory A of 100 workers is ₹720 per week. The mean wages of 30 female workers in the factory is ₹650 per week. Find out average wage of male workers in the factory.

**Solution:**

$$N_f + N_m = 100 = N$$

$$\bar{X} = 720 = \text{Mean wage of total workers} = \frac{\text{Total wage of all workers}}{N}$$

So total wage of all workers =  $720 \times N = 720 \times 100 = 72000$

$$\bar{X}_f = 650 = \text{Mean wage of total female workers} = \frac{\text{Total wage of female workers}}{N_f}$$

So total wage of female workers =  $650 \times N_f = 650 \times 30 = 19500$

So total wage of male workers =  $72000 - 19500 = 52500$

No of male workers in the factory =  $100 - 30 = 70$

So Mean wage of total male workers =  $\bar{X}_m = \frac{52500}{70} = ₹ 750$  per week

**Illustration 6.** The following table gives the life time in hours of 400 tubes of a certain make. Find the mean life time of the tubes.

Life time (hours)	Number of tubes	Life time (hours)	Number of tubes
Less than 300	0	Less than 800	265
Less than 400	20	Less than 900	324
Less than 500	60	Less than 1000	374
Less than 600	116	Less than 1100	392
Less than 700	194	Less than 1200	400

**Solution:**

As per Step Deviation method:  $\bar{X} = A + \frac{\sum fdx'}{\sum f} \times i$ . Here we have taken  $A = 749.5$

#### Calculations for Mean

Life time (in Hours)	Number of Tubes	Class Interval	Class frequency ( $f$ )	Mid value ( $m$ )	$d = m - 749.5$	$d' = d/100$	$fd'$
Less than 300	0						
Less than 400	20	300-399	20	349.5	-400	-4	-80
Less than 500	60	400-499	40	449.5	-300	-3	-120
Less than 600	116	500-599	56	549.5	-200	-2	-112
Less than 700	194	600-699	78	649.5	-100	-1	-78
Less than 800	265	700-799	71	749.5	0	0	0
Less than 900	324	800-899	59	849.5	100	1	59
Less than 1000	374	900-999	50	949.5	200	2	100
Less than 1100	392	1000-1099	18	1049.5	300	3	54
Less than 1200	400	1100-1199	8	1149.5	400	4	32
Total	—	—	400	—	—	—	-145

Mean Life time of Tubes =  $749.5 + (-)145/400 \times 100 = 713.25$  hours

**Illustration 7.** The average weight of the following distribution is 58.5 kg

Weight in Kg	50	55	60	$X+12.5$	70	Total
Number of men	1	4	2	2	1	10

Find the value of  $X$ .

**Solution:****Calculations for Mean**

Weight $w$ (kg)	Frequency ( $f$ )	$fw$
50	1	50
55	4	220
60	2	120
$X + 12.5$	2	$2X + 25$
70	1	70
Total	10	$2X + 485$

$$\text{We have Mean} = \frac{\sum fw}{\sum f} = \frac{(2X + 485)}{10}$$

By question we can write  $\frac{2X + 485}{10} = 58.5$  from where  $X$  is calculated as  $\frac{58.5 \times 10 - 485}{2} = 50$  kg.

**Illustration 8.** A appeared in three test of full marks 20, 50 and 30 marks respectively. He obtained 75% marks in the first and 60% marks in the second test. What should be his percentage of marks in the third test if his aggregate is 60%?

**Solution:**

Here the mean is obtained by the formula

$$\text{Average} = \frac{\sum_{i=1}^3 (\text{Full Marks in test } i \times \% \text{age of marks in test } i)}{20 + 50 + 30}$$

$$\text{Or, } 0.6 = \frac{20 \times 0.75 + 50 \times 0.6 + 30 \times x}{100} = \frac{15 + 30 + 30x}{100} \quad [x = \text{Percentage of marks in the 3rd test}]$$

$$\text{Or, } 60 = 45 + 30x$$

$$\text{Or, } 15 = 30x$$

$$\text{Or, } x = 50\%$$

**Illustration 9.** A certain number of salesmen were appointed in different territories and the following data were compiled from their sales reports:

Sales ('000 ₹)	No of Salesmen
4-8	11
8-12	13
12-16	16
16-20	14
20-24	?
24-28	9
28-32	17
32-36	6
36-40	4

If the average sale is believed to be ₹19, 920 find the missing information.

**Solution:**

Let the missing frequency be  $y$ .

In this problem mean is obtained by the formula

$$\bar{X} = A + \frac{\sum fd}{\sum f}$$

Here  $d$  = deviation of respective class mark derived from  $A = 22$  and the detail computations are as follows:

**Calculations for Mean**

Class Interval of Sales ('000₹)	No. of Salesmen ( $f$ )	Mid Value ( $x$ ) (000 ₹)	$d = x - 22$	$fd$
4-8	11	6	-16	-176
8-12	13	10	-12	-156
12-16	16	14	-8	-128
16-20	14	18	-4	-56
20-24	$y$	22	0	0
24-28	9	26	4	36
28-32	17	30	8	136
32-36	6	34	12	72
36-40	4	38	16	64
Total	$90 + y = \sum f$	-	-	-208

Now by question we can write

$$\bar{X} = A + \frac{\sum fd}{\sum f}$$

$$\text{Or, } 19920 = 22000 + \frac{(-)208000}{\sum f}, \text{ as sales are in ₹ thousands}$$

$$\text{Or, } 19920 - 22000 = \frac{-208000}{\sum f}$$

$$\text{Or, } -2080 = \frac{-208000}{\sum f}$$

$$\text{Or, } \sum f = 100. \text{ Or, } 90 + y = 100 \text{ Or, } y = 10.$$

So the missing frequency is 10

**Illustration 10.** The numbers 3.2, 5.8, 7.9 and 4.5 have frequencies  $x$ ,  $x + 2$ ,  $x - 3$  and  $x + 6$  respectively. If the arithmetic mean is 4.876 find the value of  $x$ .

**Solution:**

Mean is obtained by the formula:  $\bar{n} = \frac{\sum fn}{\sum f}$

So detail computations are

## Calculations for A.M

Number ( $n$ )	frequency ( $f$ )	$fn$
3.2	$x$	$3.2x$
5.8	$x + 2$	$5.8x + 11.6$
7.9	$x - 3$	$7.9x - 23.7$
4.5	$x + 6$	$4.5x + 27$
Total	$4x + 5$	$21.4x + 14.9$

So we can write

$$\text{Mean} = 4.876 = \frac{21.4x + 14.9}{4x + 5}$$

$$\text{So } x = 5$$

**Illustration 11.** The sum of deviations of a certain number of observations (ungrouped data) measured from 4 is 72 and the sum of the deviations of the observations from 7 is  $-3$ . Find the number of observations and their mean.

**Solution:**

In this problem mean is obtained by the formula

$$\bar{X} = A + \frac{\Sigma d}{n}$$

So by question we can write that

$$4 + \frac{72}{n} = 7 - \frac{3}{n}$$

$$\text{Or, } 4n + 72 = 7n - 3$$

$$\text{Or, } 3n = 75$$

$$\text{Or, } n = 25$$

$$\text{So mean} = 4 + \frac{72}{25} = 6.88$$

**Illustration 12.** The marks of a student in written and oral tests in subjects A, B and C are as follows: The written test marks are out of 70 and the oral test marks are out of 30. Find the weighted mean of the marks in written test taking the percentage of marks in oral test as weights

Subjects	Marks in Written test	Marks in Oral test
A	43	15
B	32	12
C	29	18

**Solution:****Calculations for Weighted Mean**

Oral test marks	Weight of Oral test marks out of 30 (x)	Written test marks (y)	xy
15	0.5	43	21.5
12	0.4	32	12.8
18	0.6	29	17.4
Total	1.5	—	51.7

$$\text{Mean} = \Sigma xy / \Sigma x = 51.7 / 1.5 = 34.47$$

**Illustration 13.** Calculate simple and weighted arithmetic averages from the following data:

Designation	Monthly salary (₹)	Strength of the cadre
Class I officer	1500	10
Class II officer	800	20
Subordinate Staff	500	70
Clerical Staff	250	100
Lower Staff	100	150

**Solution:**

Detail computations are:

**Calculations for Simple and Weighted A.M**

Designation	Monthly Salary (x)	Strength of the Cadre (w)	w × x
Class I Officer	1500	10	15000
Class II Officer	800	20	16000
Subordinate Staff	500	70	35000
Clerical Staff	250	100	25000
Lower Staff	100	150	15000
Total	3150	350	106000

$$\text{So simple arithmetic average} = \frac{\Sigma x}{n} = \frac{3150}{5} = ₹ 630$$

$$\text{So weighted arithmetic average} = \frac{\Sigma wx}{\Sigma w} = \frac{106000}{350} = ₹ 302.86$$

**Illustration 14.** If  $\sum_{i=1}^{10} (x-10) = 10$ . What is sample mean?

$$\begin{aligned} \text{Solution: } \sum_{i=1}^{10} x_i - \sum_{i=1}^{10} 10 &= 10 & \text{or, } \sum_{i=1}^{10} x_i - 10 \times 10 &= 10 & \text{or, } \sum_{i=1}^{10} x_i &= 10 + 100 = 110 \\ \text{mean} &= \Sigma x_i / n = 110 / 10 = 11. \end{aligned}$$

**Illustration 15.** If arithmetic mean of 1st 'n' natural numbers is  $\frac{6n}{11}$ , find 'n'.

**Solution:** As per question

$$\frac{n(n+1)}{2n} = \frac{6n}{11}$$

$$\text{or, } \frac{n+1}{2} = \frac{6n}{11} \quad \text{or, } 11n + 11 = 12n$$

$$\text{or, } 11 = 12n - 11n = n.$$

**Illustration 16.** The mean of first 3 terms is 14 and mean of rest 2 terms is 18. Find mean of all 5 terms.

**Solution:** Sum of 3 terms =  $14 \times 3 = 42$   
 Sum of next 2 terms =  $18 \times 2 = 36$   
 Sum of all 5 terms =  $42 + 36 = 78$   
 Mean of 5 terms =  $78/5 = 15.6$

**Illustration 17.** There were 50 students in a class. 10 failed whose average marks were 2.5. The total marks of class were 200. Find average marks of passed students

**Solution:** Average marks of passed students  

$$= \frac{200 - 10 \times 2.5}{40} = \frac{200 - 25}{40} = \frac{175}{40} = 4.375$$

**Properties of Arithmetic Mean:** The important properties of arithmetic mean are given below:

- (i) The sum of the deviations of the terms from the Actual mean is always zero.
- (ii) The sum of the squared deviations of the items from arithmetic mean is minimum i.e. less than the sum of the squared deviations of the items from any other value.
- (iii) If we have arithmetic mean and the number of items of two or more than two groups, we can calculate the combined average of groups.
- (iv) If the terms of a series are increased, decreased, multiplied or divided by some constant, the mean will also be increased, decreased, multiplied or divided by the same constant.
- (v) The standard error of the arithmetic mean is less than that of any other measure of central tendency.

**Merits of A.M:**

- Different interpretation by different persons not possible as AM is rigidly defined
- Easy to understand
- It takes all values in consideration
- It lends itself easily to further mathematical treatment

**Demerits of A.M:**

- It cannot be determined by inspection nor can it be located graphically
- It is affected very much by extreme values. Average of 1, 2 & 3 is 2 but average of 1, 2, 33 is extremely effected by presence of 33 and the result is 12
- It cannot be calculated if the distribution has open ended classes

**II. GEOMETRIC MEAN (g):** The geometric mean is obtained by multiplying the values of the items together and then taking it to its root corresponding to the number of items. It is denoted by 'g'. i.e.,  $g = \sqrt[n]{X_1 X_2 X_3 \dots X_n}$

(a) Individual Series:  $g = \text{Antilog} \left( \frac{\sum_{i=1}^n \log x_i}{n} \right)$

Where g is G.M,  $x_i$  is items, n is No. of terms

(b) Discrete Series:  $g = \text{Antilog} \left( \frac{\sum_{i=1}^n f_i \log x_i}{\sum_{i=1}^n f_i} \right)$

(c) Grouped Series or Continuous Series:  $g = \text{Antilog} \left( \frac{\sum f \log m}{\sum f} \right)$ , where m = Mid value of the class Interval

(d) Weighted Geometric mean:  $\text{Antilog} \left( \frac{\sum w \log x}{\sum w} \right)$

(e) Combined Geometric Mean:  $\text{Antilog} \left( \frac{n_1 \log g_1 + n_2 \log g_2}{n_1 + n_2} \right)$

- The Geometric mean is relative value and is dependent on all items
- The geometric mean is never larger than the arithmetic mean. It is rare that it may be equal to the arithmetic mean.
- The Geometric mean of the products of corresponding items is two series is equal to product of their geometric means.

**Illustration 18.** Calculate the GM of the following series of monthly income of a batch of families: 180, 250, 490, 120, 1400, 7000, 1050, 150, 360, 100, 80, 200, 500, 240.

**Solution:**

$$\text{G.M} = g = \text{Antilog} \left( \frac{\sum_{i=1}^n \log x_i}{n} \right)$$

Detail computations are

**Calculations for G.M.**

Income (x)	Log x
180	2.255273
250	2.39794
490	2.690196
120	2.079181
1400	3.146128
7000	3.845098
1050	3.021189
150	2.176091
360	2.556303

100	2,000000
80	1.90309
200	2.30103
500	2.69897
240	2.380211
Total	35.4507

$$\text{So, } g = \text{Antilog} \left( \frac{35.4507}{14} \right) = 340.6$$

**Illustration 19.** Calculate the average income per head from the data given below using GM:

Class of People	Number of Families	Annual Income/head (₹'00)
Landlords	2	5000
Cultivators	100	400
Landless Labourers	50	200
Money lenders	4	3750
School teachers	6	3000
Shop-Keepers	8	750
Carpenters	6	600
Weavers	10	300

**Solution:**

$$\text{For Discrete Series: G.M} = g = \text{Antilog} \left( \frac{\sum_{i=1}^n f_i \log x_i}{\sum f_i} \right)$$

Detail computations are as follows:

#### Calculations for G.M.

Class of people	No. of Families ( $f$ )	Income ( $x$ )	$\log x$	$f \log x$
Landlords	2	5000	3.69897	7.39794
Cultivators	100	400	2.60206	260.206
Landless Labourers	50	200	2.30103	115.0515
Money Lenders	4	3750	3.574031	14.29613
School Teaches	6	3000	3.477121	20.86273
Shop Keepers	8	750	2.875061	23.00049
Carpenters	6	600	2.778151	16.66891
Weavers	10	300	2.477121	24.77121
Total	186	—	—	482.2549

$$g = \text{Antilog} \left( \frac{482.2549}{186} \right) = 391.54 \text{ (₹'00)}$$

**Illustration 20.** The weighted geometric mean of four numbers 8, 25, 19 and 28 is 22.15. If the weights of the first three numbers are 3, 5, 7 respectively, find the weight of the fourth number.

**Solution:**

Weighted Geometric mean:  $Antilog\left(\frac{\sum w \log X}{\sum w}\right) = 22.15$

So  $\left(\frac{\sum w \log X}{\sum w}\right) = \text{Log}(22.15) = 1.3454$

Detail computations are as follows:

**Calculations of Weighted G.M.**

$X$	$w$	$\log X$	$w \log X$
8	3	0.90309	2.71
25	5	1.39794	6.99
19	7	1.278754	8.95
28	$w$	1.447158	1.45w
Total	$15 + w$		$18.65 + 1.45w$

So  $\frac{18.65 + 1.45w}{15 + w} = 1.3454$  So  $w = 15$  (approx)

**Illustration 21.** If A.M and G.M of two observations are 5 and 4, Then find two observations.

**Solution:**  $\frac{a+b}{2} = 5$  or,  $a + b = 10$  -----(1)

$\sqrt{ab} = 4$  or,  $ab = 16$  -----(2)

$(a-b)^2 = (a+b)^2 - 4ab = 10^2 - 4 \times 16 = 36$  or,  $a-b = 6$ . -----(3)

Solving 1 & 3  $a = 8$  &  $b = 2$ .

**Illustration 22.** If G.M of  $x$  is 5 and G.M of  $y$  is 6, then find G.M of  $xy$ .

**Solution:** G.M of  $xy = \text{G.M of } x \times \text{G.M of } y = 5 \times 6 = 30$

**Illustration 23.** If G.M of  $x$  and  $y$  are 5 & 10,

find G.M of  $\frac{y}{x}$ .

**Solution:** G.M of  $\frac{y}{x} = \frac{\text{G.M. of } y}{\text{G.M. of } x} = \frac{10}{5} = 2$

**Illustration 24.** Find G.M of  $a, a^2, \dots, a^n$ .

**Solution:** G.M  $= (a, a^2, \dots, a^n)^{\frac{1}{n}} = (a^{1+2+\dots+n})^{\frac{1}{n}}$   
 $= a^{\frac{n(n+1)}{2} \times \frac{1}{n}} = a^{\frac{n+1}{2}}$

**Merits of G.M:**

- Different interpretation by different persons not possible as GM is rigidly defined
- It takes all value in consideration
- Unlike A.M, it is not affected much by the presence of extremely small or large values
- It lends itself easily to further mathematical treatment

**Demerits of G.M:**

- It is difficult to understand
- It is difficult to calculate particularly when the items are very large or when there is graphical data
- It becomes imaginary if any of the observations is negative and GM is zero if any of the observation is zero

**III. HARMONIC MEAN:**

Harmonic mean is the reciprocal of the arithmetic mean

(a) For ungrouped data:  $H.M = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}$ , where  $x_i$  = Values of the items and  $n$  = No. of terms

(b) For data with frequency:  $H.M = \frac{\sum f_i}{\sum_{i=1}^n \frac{f_i}{x_i}}$ , where  $x_i$  = Values of the items (for ungrouped data) or mid value of the class (for grouped data)

(c) Weighted  $H.M = \frac{\sum w}{\sum \frac{w}{x}}$ , where  $w$  = Weights and  $x$  = Values of the items

**Illustration 25.** Find the H.M of 3, 6, 12, 24 whose weights are 6, 2, 4, 8 respectively.

**Solution:**

$$H.M = \frac{\sum w}{\sum \frac{w}{x}} = \frac{6+2+4+8}{\frac{6}{3} + \frac{2}{6} + \frac{4}{12} + \frac{8}{24}} = \frac{20}{2 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \frac{20}{2+1} = 6\frac{2}{3}$$

**Illustration 26.** The marks secured by some students of a class are given below. Calculate the harmonic mean.

Marks	20	21	22	23	24	25
No of students	4	2	7	1	3	1

**Solution:****Calculations for H.M**

Marks ( $X$ )	No of Students ( $f$ )	Reciprocal of $X$	$f \times$ Reciprocal of $X$
20	4	0.05000	0.20000
21	2	0.04762	0.09524
22	7	0.04545	0.31815
23	1	0.04348	0.04348
24	3	0.04167	0.12501
25	1	0.04000	0.04000
Total	18	–	0.82188

$$\text{H.M} = \frac{\sum f}{\sum \frac{f}{x}} = \frac{18}{0.82188} = 21.90$$

**Merits of H.M:**

- Different interpretation by different persons not possible as HM is rigidly defined
- It takes all value in consideration
- It is most suitable average when it is desired to give greater weight to smaller observations and less weight to the larger ones
- It lends itself easily to further mathematical treatment

**Demerits of H.M:**

- It is difficult to understand
- It is difficult to compute
- It is only a summary figure and may not be the actual item in the series

**Relationship among the different Averages:**

- (a) In any distribution where the original items differ in size, then either the values of  $A.M > G.M > H.M$  (or)  $H.M < G.M < A.M$ . In case all items are identical then  $A.M. = G.M = H.M$
- (b) Let  $X, Y$  be two numbers then:  $A.M = \frac{X+Y}{2}$ ,  $G.M = \sqrt{X \cdot Y}$ ,  $H.M = \frac{2XY}{X+Y}$

**IV. MEDIAN AND OTHER POSITIONAL MEASURES:** Median is denoted by  $M_d$ . It is a positional measurement. Median divides a series into two equal parts. i.e., the middle most item is called median. It is also indicated as  $M_d$ .

(a) **Individual Series:** First of all the terms are to be arranged in ascending (or) descending order.

(1) When number of terms is odd then

Median ( $M_d$ ) =  $\frac{n+1}{2}$  th item

Where  $M_d$  is Median,  $n$  is No. of terms in the given series.

(2) When number of terms is even, then

Median ( $M_d$ ) = Average of  $\left(\frac{n}{2}\right)$  &  $\left(\frac{n}{2}+1\right)$  th terms.

(b) **Discrete Series with frequency:** Median ( $M_d$ ) is the  $\left(\frac{N+1}{2}\right)$  th term. Here  $N = \Sigma f =$  Total frequency

(c) **Grouped data** (Less than type i.e. When cumulative frequency is calculated from above)

$$M_d = l_1 + \frac{\frac{N}{2} - C}{f} \times h$$

Where  $l_1$  = Lower boundary of the median class

$N = \Sigma f$

$C'$  = cumulative frequency just preceding the median class.

$f$  = simple frequency of the median class

$h$  = width of the median class

(d) **Grouped data** (Greater than type i.e. When cumulative frequency is calculated from below)

$$M_d = l_2 - \frac{\frac{N}{2} - C''}{f} \times h$$

Where  $l_2$  = Upper boundary of the median class

$N = \Sigma f$

$C''$  = cumulative frequency just succeeding the median class.

$f$  = simple frequency of the median class

$h$  = width of the median class

**Illustration 27.** Find Median from following data:

17, 19, 21, 13, 16, 18, 24, 22, 20

**Solution:**

Arranging the data in ascending order we get: 13, 16, 17, 18, 19, 20, 21, 22, 24

No of observations in the data  $n = 9$

So Median  $M_d = \frac{n+1}{2} = \frac{9+1}{2} = 5$  th item i.e. 19

**Illustration 28.** Find Median from following data:

17, 19, 21, 13, 16, 18, 24, 22, 20, 26

**Solution:**

Arranging the data in ascending order we get: 13, 16, 17, 18, 19, 20, 21, 22, 24, 26

No of observation in the data  $n = 10$

So Median  $M_d = \text{Average of } \left(\frac{n}{2}\right) \text{ \& } \left(\frac{n}{2} + 1\right) \text{ th items i.e 5th \& 6th items} = \frac{19 + 20}{2} = 19.5$

**Illustration 29.** Compute Median for the following data:

X	10	20	30	40	50	60	70
f	4	7	21	34	25	12	3

**Solution:**

**Calculations for Median**

	X	f	C.f.
	10	4	4
	20	7	11
	30	21	32
Median →	40	34	66 ← 53.5
	50	25	91
	60	12	103
	70	3	106 = n

$$\text{Median} = M_d = \frac{n+1}{2} \text{th term} = \frac{106+1}{2} = 53.5 \text{th term} = 40$$

[From the above table all the values between 33rd and 66th items are 40. Thus 53.5th term should be 40]

**Illustration 30.** Calculate Median from the following data.

Class	4-8	8-20	20-28	28-40	40-60	60-72
f	7	12	42	56	39	22

**Solution:**

Here  $N/2 = 178/2 = 89$  i.e 89th value is the Median. Hence it is a value in the class 28 – 40.

## Calculations for Median

Class	Frequency ( $f$ )	C.f (less than type)
4-8	7	7
8-20	12	19
20-28	42	61
28-40	56	117 Median Class
40-60	39	156
60-72	22	178 = N

So

$$M_d = l_1 + \frac{\frac{N}{2} - C}{f} \times h$$

Where  $l_1$  = Lower boundary of the median class = 28

$$N = \Sigma f = 178$$

$C$  = cumulative frequency just preceding the median class = 61

$f$  = simple frequency of the median class = 56

$h$  = width of the median class =  $40 - 28 = 12$

$$M_d = l_1 + \frac{\frac{N}{2} - C}{f} \times h = 28 + \frac{\frac{178}{2} - 61}{56} \times 12 = 34$$

OR,

$$M_d = l_2 - \frac{\frac{N}{2} - C''}{f} \times h$$

Where  $l_2$  = Upper boundary of the median class = 40

$$N = \Sigma f = 178$$

$C''$  = cumulative frequency just succeeding the median class = 61

$f$  = simple frequency of the median class = 56

$h$  = width of the median class = 12

Class	Frequency	C.f (More than type)
4-8	7	178 = N
8-20	12	171
20-28	42	159
28-40	56	117 Median Class
40-60	39	61
60-72	22	22

So

$$M_d = l_2 - \frac{\frac{N}{2} - C''}{f} \times h = 40 - \frac{\frac{178}{2} - 61}{56} \times 12 = 34$$

Graphical method for locating median:

- This can be determined by drawing *Ogive* i.e. cumulative frequency curves.
- Cumulative frequency curves are of two types—More than c.f curve & less than c.f. curve
- These curves are drawn against class boundaries.
- The intersection of these two curves indicate the Median value on the X axis

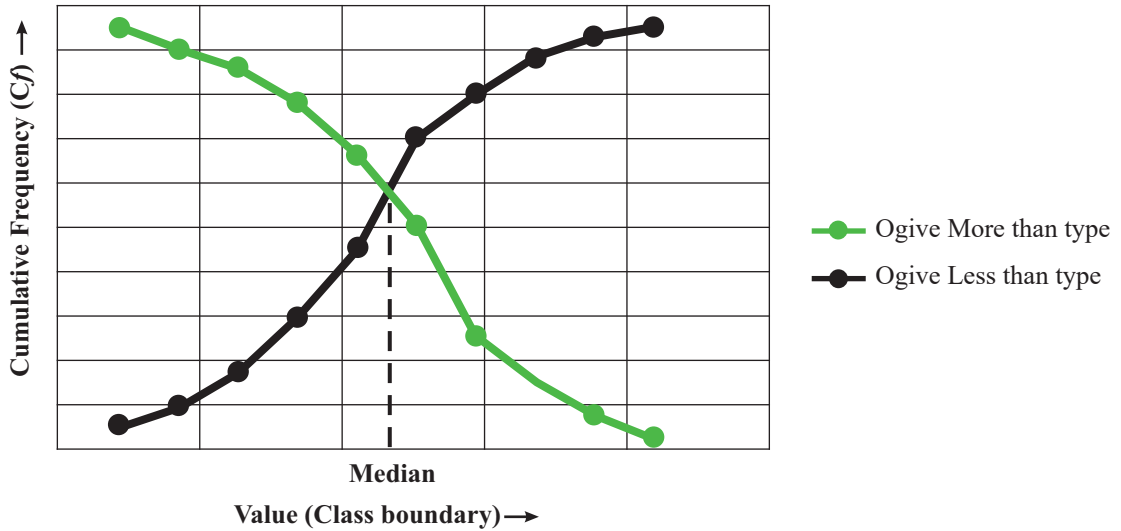
For example let us consider the following data

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No of Students	5	4	8	12	16	25	10	8	5	2

To draw the two ogives we first find less than type c.f and more than type c.f against respective class boundary and the detail computations are

Class	Frequency	Class Boundary	Cf(More thantype)	Cf(Less than type)
0-10	5	10	95	5
10-20	4	20	90	9
20-30	8	30	86	17
30-40	12	40	78	29
40-50	16	50	66	45
50-60	25	60	50	70
60-70	10	70	25	80
70-80	8	80	15	88
80-90	5	90	7	93
90-100	2	100	2	95

And the ogives will be as follows: (Class boundaries on x axis and c.f. or y axis)



Median is obtained by drawing an ordinate at the point of intersection of the two Ogives, as shown. The 'Value' corresponding to this ordinate is the Median.

Median divides the distribution into two equal parts. So we take  $\frac{n+1}{2}$  for finding out median of ungrouped data. Similarly for ungrouped data we can find out –

(A) **Quartile:** Divides the distribution into 4 equal parts:

$$1^{\text{st}} \text{ quartile} = Q_1 = \frac{n+1}{4} \text{ th term}$$

$$2^{\text{nd}} \text{ quartile} = Q_2 = 2 \times \frac{n+1}{4} \text{ th term}$$

$$3^{\text{rd}} \text{ quartile} = Q_3 = 3 \times \frac{n+1}{4} \text{ th term}$$

(B) **Decile:** Divides the distribution into 10 equal parts:

$$1^{\text{st}} \text{ decile} = D_1 = \frac{n+1}{10} \text{ th term}$$

$$2^{\text{nd}} \text{ decile} = D_2 = 2 \times \frac{n+1}{10} \text{ th term}$$

.....

$$5^{\text{th}} \text{ decile} = D_5 = 5 \times \frac{n+1}{10} \text{ th term}$$

.....

$$9^{\text{th}} \text{ decile} = D_9 = 9 \times \frac{n+1}{10} \text{ th term}$$

(C) **Percentile:** Divides the distribution into 100 equal parts:

$$1^{\text{st}} \text{ Percentile} = P_1 = \frac{n+1}{100} \text{ th term}$$

$$2^{\text{nd}} \text{ Percentile} = P_2 = 2 \times \frac{n+1}{100} \text{ th term}$$

.....

$$50^{\text{st}} \text{ Percentile} = P_{50} = 50 \times \frac{n+1}{100} \text{ th term}$$

.....

$$99^{\text{th}} \text{ Percentile} = P_{99} = 99 \times \frac{n+1}{100} \text{ th term}$$

From above it is clear

$$\begin{aligned} 2^{\text{nd}} \text{ quartile} = Q_2 &= 2 \times \frac{n+1}{4} \text{ th term} = 5^{\text{th}} \text{ decile} = D_5 = 5 \times \frac{n+1}{10} \text{ th term} = 50^{\text{th}} \text{ decile} = P_{50} \\ &= 50 \times \frac{n+1}{100} \text{ th term} = \frac{n+1}{2} \text{ th term} = \text{median} \end{aligned}$$

Similarly for grouped data we can write

$$M_d = l_1 + \frac{\frac{N}{2} - C}{f} \times h$$

$$1^{\text{st}} \text{ quartile} = Q_1 = l_1 + \frac{\frac{N}{4} - C}{f} \times h$$

$$2^{\text{nd}} \text{ quartile} = Q_2 = l_1 + \frac{2 \times \frac{N}{4} - C}{f} \times h$$

$$3^{\text{rd}} \text{ quartile} = Q_3 = l_1 + \frac{3 \times \frac{N}{4} - C}{f} \times h$$

Similarly Deciles and Percentiles can be calculated.

$$N = \sum f$$

$C = C.f$  just preceding to respective quartile class,

$l_1$  = real lower limit of respective quartile class,

$f$  = simple frequency of respective quartile class,

$h$  = width of quartile class

Similarly Deciles and Percentiles can be calculated.

**Illustration 31.** The Median of  $x, \frac{x}{2}, \frac{x}{3}, \frac{x}{4}$  is 5.

Find 'x'.

**Solution:** Median =  $\frac{1}{2} \left( \frac{x}{2} + \frac{x}{3} \right) = 5$

or,  $5x = 60$

or,  $x = 12$

**Merits of Median:**

- It is not at all affected by the items on the extremes
- It can be computed for the distributions which have open ended classes
- The value of the median can be located graphically
- The median is centrally located. The absolute sum of the deviations of the individual values from the median is always the minimum

**Demerits of Median:**

- In case of an even number of observations, median cannot be determined exactly
- It does not lend itself to algebraic treatment in a satisfactory manner
- It is unsuitable if it is desired to give greater importance to large or small values

**Illustration 32.** From the following grouped data find Median, 2<sup>nd</sup> Quartile, 5<sup>th</sup> Decile and 50<sup>th</sup> percentile

Weekly Wages (₹)	30-32	32-34	34-36	36-38	38-40	40-42	42-44	44-46	46-48	48-50
Frequency	3	8	24	31	50	61	38	21	12	2

**Solution:**

Detail less than type cummulation is as follows:

Wages	Frequency	c.f (less than type)
30-32	3	3
32-34	8	11
34-36	24	35
36-38	31	66
38-40	50	116
40-42	61	177 Median Class
42-44	38	215
44-46	21	236
46-48	12	248
48-50	2	250 = N

Since  $\frac{N}{2} = \frac{250}{2} = 125$ , the median is 125th value. Now class 38-40 contains only upto 116th value. Any value having rank greater than 116 but less than or inclusive 177 falls in the class 40-42. So median class is marked against 40-42

$$\text{Median} = M_d = l_1 + \frac{\frac{N}{2} - C}{f} \times h = 40 + \frac{125 - 116}{61} \times 2 = 40.295$$

$$2^{\text{nd}} \text{ quartile} = Q_2 = l_1 + \frac{2 \times \frac{N}{4} - C}{f} \times h = 40 + \frac{2 \times \frac{250}{4} - 116}{61} \times 2 = 40.295$$

$$5^{\text{th}} \text{ decile} = D_5 = l_1 + \frac{5 \times \frac{N}{10} - C}{f} \times h = 40 + \frac{5 \times \frac{250}{10} - 116}{61} \times 2 = 40.295$$

$$50^{\text{th}} \text{ percentile} = P_{50} = l_1 + \frac{50 \times \frac{N}{100} - C}{f} \times h = 40 + \frac{50 \times \frac{250}{100} - 116}{61} \times 2 = 40.295$$

[From the above computations we can conclude, Median = 2nd Quartile = 5th Decile = 50th Percentile]

**Illustration 33.** Refer to illustration 22. Compute 6<sup>th</sup> Decile and 88<sup>th</sup> Percentile

**Solution:**

$$6^{\text{th}} \text{ decile} = D_6 = l_1 + \frac{6 \times \frac{N}{10} - C}{f} \times h = 40 + \frac{6 \times \frac{250}{10} - 116}{61} \times 2 = 41.115$$

$[6 \times \frac{250}{10} = 6 \times \frac{250}{10} = 150^{\text{th}}$  value is the 6th Decile. Thus the class 40 – 42 is the 6th Decile class]

$$88^{\text{th}} \text{ percentile} = P_{88} = l_1 + \frac{88 \times \frac{N}{100} - C}{f} \times h = 44 + \frac{88 \times \frac{250}{100} - 215}{21} \times 2 = 44.476$$

$[88 \times \frac{N}{100} = 88 \times \frac{250}{100} = 220^{\text{th}}$  value is the 88<sup>th</sup> Percentile which belongs to the class 44 – 46]

**Calculations for 6<sup>th</sup> Decile and 88<sup>th</sup> Percentile**

Wages	Frequency	c.f
30-32	3	3
32-34	8	11
34-36	24	35
36-38	31	66
38-40	50	116
40-42	61	177 ← 6 <sup>th</sup> Decile Class
42-44	38	215
44-46	21	236 ← 88 <sup>th</sup> Percentile Class
46-48	12	248
48-50	2	250

**V. MODE:** It is denoted by 'M<sub>0</sub>'. Mode may be defined as the value that occurs most frequently in a statistical distribution.

(i) **Individual Series and Discrete Series:** The terms are arranged in any order, Ascending or Descending. If each term of the series is occurring once, then there is no mode, otherwise the value or values which occurs maximum times is/are known as Mode/Modes.

(ii) **Grouped data:** 
$$M_o = l_1 + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

Where  $l_1$  = Lower boundary of the modal class

$f_0$  = Simple frequency of the modal class

$f_1$  = Simple frequency of the class just preceding the modal class

$f_2$  = Simple frequency of the class just succeeding the modal class

$h$  = width of the modal class

**Illustration 34.** Find Mode from the following data: 12, 14, 16, 18, 26, 16, 20, 16, 11, 12, 16, 16, 20, 24

**Solution:**

In the above data

12 appeared twice;

16 appeared 5 times;

20 appeared twice;

Other terms only once

So mode is 16 as it appeared maximum times

**Illustration 35.** Find Mode from the following data:

Class	0-10	10-20	20-30	30-40	40-50	50-60	60-70
f	4	13	21	44	33	22	7

**Solution:**

**Table showing Frequency Distribution**

	Class	Frequency (f)
Modal Class →	0-10	4
	10-20	13
	20-30	21 = $f_1$
	30-40	44 = $f_0$
	40-50	33 = $f_2$
	50-60	22
	60-70	7

So

$$\text{Mode} = M_o = l_1 + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h = 30 + \frac{44 - 21}{2 \times 44 - 21 - 33} \times 10 = 36.765$$

**Illustration 36.**

If  $y=12+1.5x$  and mode of  $x$  is 15.

What is the mode of  $y$  ?

**Solution:**  $y = 12+1.5x$

Mode of  $y = 12+1.5$  (mode of  $x$ )

$$= 12+1.5 \times 15 = 34.5$$

**Illustration 37.**

Find modal letter of the word ‘STATISTICS’.

**Solution:** Both S & T.

**Mean, Median and Mode – Their Relation:**

$$\text{Mean} - \text{Mode} = 3 (\text{Mean} - \text{Median})$$

This formula was expressed by Karl Pearson. It is called Bi-mode.

**Illustration 38.**

Compute the arithmetic average if the changes in the prices of securities in a certain stock exchange are as follows:

Class	Frequency	Class	Frequency
10% & over	54	-2% to 0%	149
8% to 10%	30	-4% to -2%	100
6% to 8%	50	-6% to -4%	74
4% to 6%	71	-8% to -6%	31
2% to 4%	119	-10% and under	51
0% to 2%	346	Total frequency	1075

**Solution:**

Classes are open ended type. So for open ended class AM can be found out by the relation

$$\text{Mean} - \text{Mode} = 3 (\text{Mean} - \text{Median})$$

Here  $\frac{N}{2} = \frac{1075}{2} = 537.5$ . So Median is the 537.5th value which belongs to the class 0 to 2.

Median is calculated from the formula

$$M_d = l_1 + \frac{\frac{N}{2} - C}{f} \times h$$

$$M_d = 0 + \frac{537.5 - 405}{346} \times 2 = 0.7659$$

As the class 0 - 2 has maximum frequency 346, it is the modal class.

Mode is calculated from the formula

$$M_o = l_1 + \frac{f_0 - f_1}{(f_0 - f_2) + (f_0 - f_1)} \times h$$

$$M_o = 0 + \frac{346 - 149}{(346 - 119) + (346 - 149)} \times 2 = 0.9292$$

So Mean = 0.9292 = 3 × (Mean - 0.7659) Or, 2 Mean = 1.3685. So Mean = 0.684 Details computation for less than cumulative frequencies are as follows:

#### Calculations for Median and Mode

Class Interval for changes in the price of the securities	Frequency	Cumulative frequency (less than)
below (-10)	51	51
8 to 6	31	82
6 to 4	74	156
4 to 2	100	256
2 to 0	149 = $f_1$	405
0 to 2	346 = $f_0$	751 ← $\frac{N}{2} = 537.5$
2 to 4	119 = $f_2$	870
4 to 6	71	941
6 to 8	50	991
8 to 10	30	1021
10 to 12	54	1075 = N
Total	1075	

**Uses of various Averages:** The use or application of a particular average depends upon the purpose of the investigation. Some of the cases of different averages are as follows:

- (a) **Arithmetic Mean:** Arithmetic mean is considered to be an ideal average. It is frequently used in all the aspects of life. It possesses many mathematical properties and due to this it is of immense utility in further statistical analysis. In economic analysis arithmetic mean is used extensively to calculate average production,

average wage, average cost, per capita income, exports, imports, consumption, prices etc. When different items of a series have different relative importance, then weighted arithmetic mean is used.

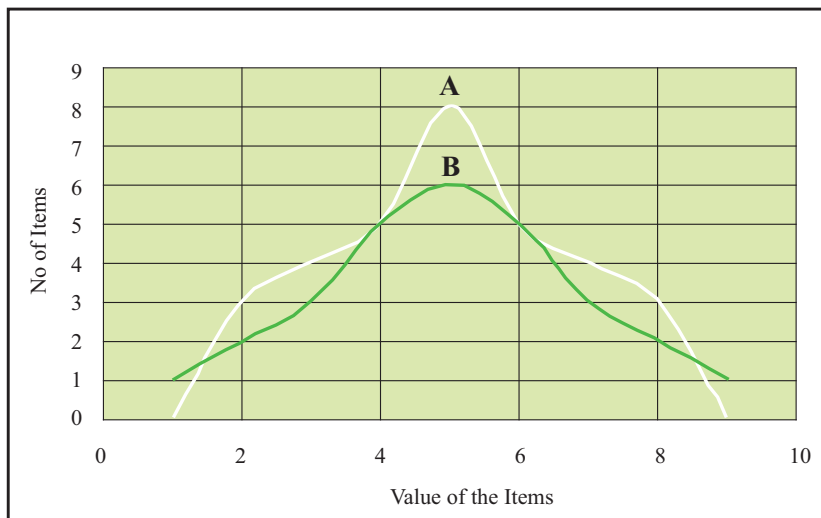
- (b) **Geometric Mean:** Use of Geometric mean is important in a series having items of wide dispersion. It is used in the construction of Index Number. The averages of proportions, percentages and compound rates are computed by geometric mean. The growth of population is measured in it as population increases in geometric progression.
- (c) **Harmonic Mean:** Harmonic mean is applied in the problems where small items must get more relative importance than the large ones. It is useful in cases where time, speed, values given in quantities, rate and prices are involved. But in practice, it has little applicability.
- (d) **Median and Partition values:** Median and partition values are positional measures of central tendency. These are mainly used in the qualitative cases like honesty, intelligence, ability etc. In the distributions which are positively skewed, median is a more suitable average. These are also suitable for the problems of distribution of income, wealth, investment etc.
- (e) **Mode:** Mode is also positional average. Its applicability to daily problems is increasing. Mode is used to calculate the 'modal size of a collar', modal size of shoes,' or 'modal size of ready-made garments' etc. It is also used in the sciences of Biology, Meteorology, Business and Industry.

# Range, Quartiles and Quartile Deviation

## 5.2

### Dispersion:

Follow the diagrammatic representation of two frequency distribution A and B. Both has same central tendency (mean) located as shown at A and B. But a close observation of the diagram indicates that individual values fall away from the central value (A and B) in different way. The drawn curves have different variability.

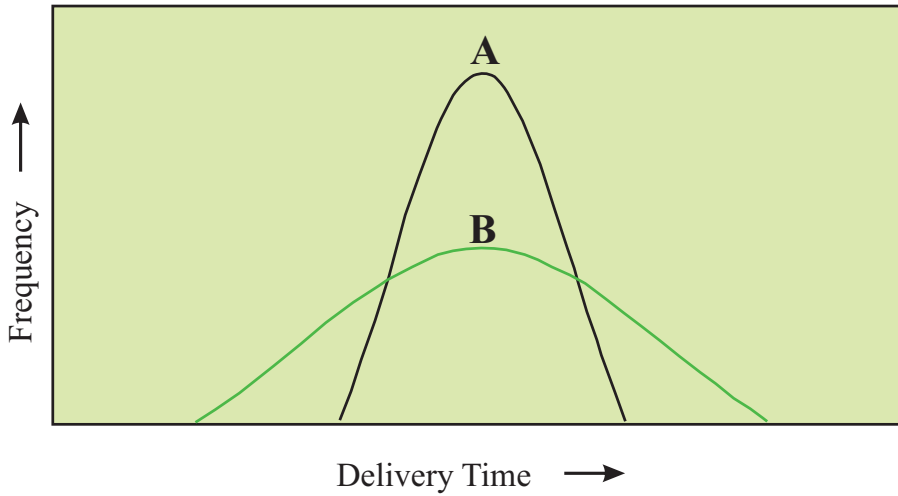


The measure of this variability of individual items with respect to central tendency is called DISPERSION.

Why knowledge on Dispersion is Important?

- It gives additional information on the data and enables us to judge the reliability of our measure of central tendency. If individual items are widely spread with respect to the measure of central tendency measured (mean/median/mode) then the central tendency is considered as less representative of the distribution as a whole.
- There are problems peculiar to wide spread data items. So dispersion facilitates us to locate those data before tackling those peculiar problems.
- On many occasions we require comparison of various samples of distribution in order to eliminate widely fluctuated data with an ultimate objective to mitigate risks. More fluctuation means more risks. We need help of dispersion on those occasions.

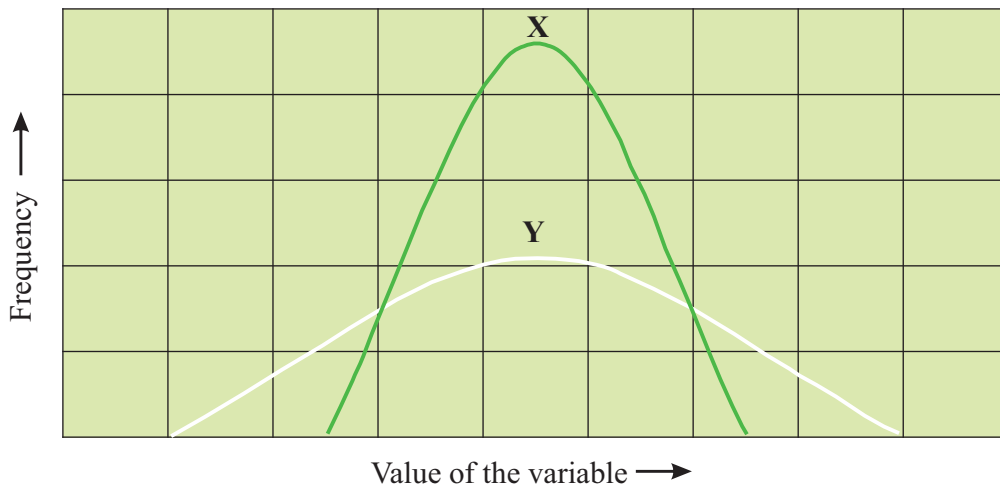
**Illustration 39.** A firm using two different methods to ship orders to its customers found the following distributions of delivery time for the two methods, based on past records. From available evidence, which shipment method you recommend?



**Solution:**

A is more preferable as B has wider fluctuation

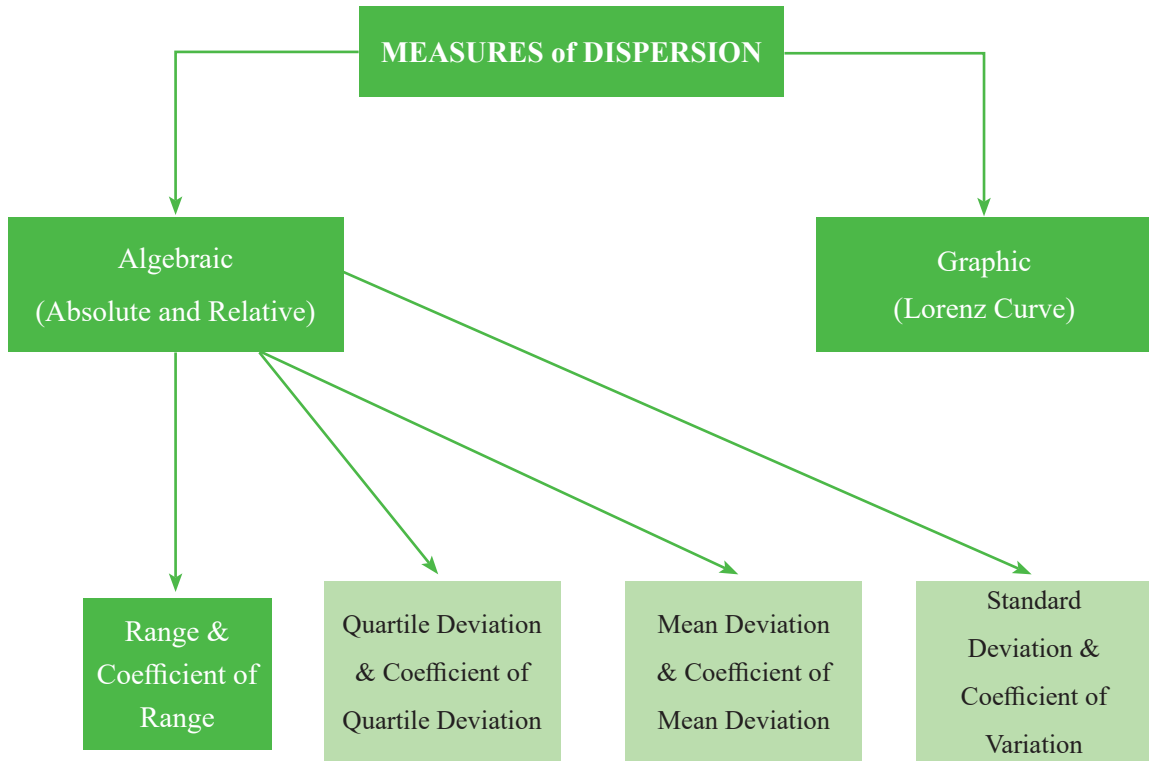
**Illustration 40.** For which one of the following distributions is the mean more representative of the data as a whole? Why?



**Solution:**

X, because the values tend to cluster more around the mean.

So DISPERSION is the degree of the scatter or variation of the variables about a central value.



**Dispersion** can be measured by (1) Distance Measures (2) Average Deviation Measures

**Distance measure:**

**Range:**

Range is the difference between the highest and lowest observed values.

*Range = Value of highest observation – Value of lowest observation*

**Illustration 41.** Marks obtained by 12 students in the recently concluded examination is as follows:

Student	Marks
1	52
2	13
3	35
4	46
5	89
6	10
7	29
8	95
9	66
10	72
11	48
12	71

From the above table it can be noted that Highest observed value is 95 whereas Lowest observed value is 10.

So Range of marks =  $95 - 10 = 85$

**Quasi range:** it refers to the difference between the values leaving the extreme values. If we leave the two extreme values it will be

$$\text{Quasi Range} = 2\text{nd highest observation} - 2\text{nd lowest observation} = 89 - 13 = 76 \text{ (in above case)}$$

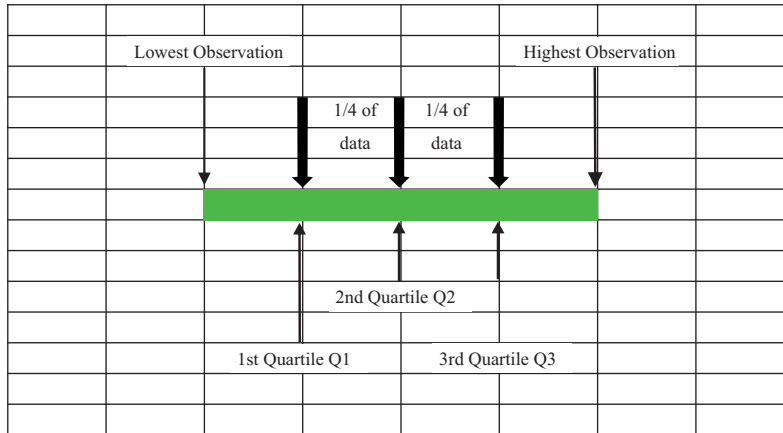
- Range is easy to understand but its usefulness as a measure of dispersion is limited;
- It considers only highest and lowest values and ignores all others. So it is highly influenced by extreme values;
- Since it measures only two values, the range is likely to change drastically from one sample to the next in a given population;
- Open ended distribution has no range (Open ended example: Data  $< X$  or Data  $> Y$ , where X and Y are any values ) ;

**Coefficient of Range:**

$$\text{Coefficient of range} = \frac{\text{highest value} - \text{lowest value}}{\text{highest value} + \text{lowest value}} = \frac{95 - 10}{95 + 10} = 0.810$$

**Interquartile Range and Quartile Deviation:**

Interquartile Range =  $Q_3 - Q_1$ , where  $Q_3$  is the third Quartile &  $Q_1$  is the first quartile



The above figure shows the concept of the interquartile range graphically.

One half of the interquartile range is a measure called the *Quartile deviation* =  $\frac{Q_3 - Q_1}{2}$ . It is also called Semi Inter Quartile Range.

The quartile deviation, then measures the average range of one-fourth of the data.

For our sample example let us calculate interquartile range and quartile deviation:

We begin by dividing the items into 4 equal parts as we have done in the following table:

FIRST FOURTH	SECOND FOURTH	THIRD FOURTH	LAST FOURTH
10	35	52	72
13	46	66	89
29 ⇐ Q1	48	71 ⇐ Q3	95

$$\text{Interquartile range} = Q_3 - Q_1 = 71 - 29 = 42$$

$$\text{Quartile deviation} = \frac{Q_3 - Q_1}{2} = \frac{42}{2} = 21$$

Like the range the interquartile range and the quartile deviation are based on only two values from the data set. Although they are more complicated to calculate than the range, they avoid extreme values by using only the middle half of the data. So effects of extreme data are avoided here.

#### Coefficient of Quartile Deviation:

$$\text{Coefficient of quartie deviation} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

#### Merits:

- Easy to calculate and simple to understand
- It can be easily computed from distribution with open ended classes
- It is useful when it is desired to know variability in the central half of data.

**Demerits:**

- It ignores completely 50% of the data
- It does not lend itself to further mathematical treatment
- It is in fact not a measure of dispersion as it really does not show scatter around the average

**Illustration 42.** For the following data, compute the a) Interquartile range; b) Quartile deviation;

97	72	87	57	39	81	70	84	93	79
84	81	65	97	75	72	84	46	94	77

**Solution:**

First arrange the data in ascending order and divide the data in four equal parts as given in the following table:

First Fourth	Second Fourth	Third Fourth	Last Fourth
39	72	81	87
46	72	81	93
57	75	84	94
65	77	84	97
70	79	84 = $Q_3$	97

(a) *Interquartile range* =  $Q_3 - Q_1 = 84 - 70 = 14$

(b) *Quartile deviation* =  $\frac{Q_3 - Q_1}{2} = \frac{14}{2} = 7$

**Average Deviation Measures:**

Under these two important measures are *Variance* and *Standard Deviation*

For a clear understanding on these two measures let us first discuss Average Absolute Deviation.

**Average Absolute Deviation or Mean Deviation:**

To compute this, steps are—

- Find out the mean of the given samples;
- Determine the absolute value of the difference between each item in the data set and the mean;
- Add all these differences together and divide by the total number of items in the sample;

For ungrouped data:

Average Absolute Deviation =  $\frac{\sum |x - \mu|}{N}$ , for a population.

Average Absolute Deviation =  $\frac{\sum |x - \bar{x}|}{n}$ , for a sample.

Where:

$x$  = the item or observation;

$\mu$  = the population mean;

$N$  = number of items in the population;

$\bar{x}$  = sample mean;

$n$  = no of items in the sample;

These are the average absolute deviation from Mean. Actually we can measure the average absolute deviation from Mean, Median or Mode. i.e.

$$\text{Average Absolute Deviation} = \frac{\sum |(x - A)|}{n}$$

Where A can be Mean =  $\bar{X}$ , Median =  $M_d$  or Mode =  $M_o$ .

**Illustration 43.** Compute the average absolute deviation from the following data table which represents payments received by 12 officers in a company on account of local conveyance during a period.

Payment in ₹	863	903	957	1041	1138	1204	1354	1624	1698	1745	1802	1883
--------------	-----	-----	-----	------	------	------	------	------	------	------	------	------

**Solution:**

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n} \\ &= \frac{863 + 903 + 957 + 1041 + 1138 + 1204 + 1354 + 1624 + 1698 + 1745 + 1802 + 1883}{12} \\ &= \frac{16212}{12} = 1351\end{aligned}$$

#### Calculations for Mean Absolute Deviation

Observation	Deviation from Mean ( $x - \text{mean}$ ) = ( $x - \bar{x}$ ) = $x - 1351$	Absolute deviation from Mean $ (x - \text{Mean})  =  x - \bar{x} $
863	-488	488
903	-448	448
957	-394	394
1041	-310	310
1138	-213	213
1204	-147	147
1354	3	3
1624	273	273
1698	347	347
1745	394	394
1802	451	451
1883	532	532
		$\sum  x - \bar{x}  = 4000$

$$\text{Average Absolute Deviation about Mean} = \frac{\sum |(x - \bar{x})|}{n} = \frac{4000}{12} = ₹ 333.3$$

This average absolute deviation is a better measure of dispersion than the range (we have already discussed) because it takes every observation into account. It gives weightage to each observation equally and indicates how far on average each observation lies from the mean.

For Grouped data:

$$\text{Average Absolute Deviation} = \frac{\sum f|(x - A)|}{\sum f}$$

Where A can be Mean =  $\bar{X}$ , Median =  $M_d$  or Mode =  $M_0$  &  $f$  = frequency

#### Merits:

- Mean deviation is easy to understand.
- It is based on all the items of the series.
- It is less affected by extreme observations when compared with Standard deviation.

#### Demerits:

- It is the arithmetic mean of the absolute values of the deviation. It ignores the positive and negative signs of deviations. This weakness creates the demand for a more reliable measure of dispersion.
- It is not amenable for further algebraic treatment.

# Standard Deviation

## 5.3

### Population Variance:

If we shift our calculation from samples to population we will get population variance symbolized by  $\sigma^2$  (sigma squared).

The population variance is similar to an average absolute deviation computed for an entire population. It is the Mean Square deviation about Mean.

$$\sigma^2 = \frac{\sum(x - \mu)^2}{N} = \frac{\sum x^2}{N} - \mu^2$$

Where:

$\sigma^2$  = population variance;

$x$  = the item or observation;

$\mu$  = population mean;

$N$  = total number of items in the population;

$\sigma^2$  has the unit which is square of the units of the data. These types of units are not clearly interpreted. To avoid this problem instead of variance we have another measure called Standard Deviation.

### Population Standard Deviation:

Population standard deviation is simply the square root of the population variance.

The standard deviation is the square root of the average of the squared distances of the observations from the mean.

$$= \sqrt{\sigma^2} = \sqrt{\frac{\sum(x - \mu)^2}{N}} = \sqrt{\frac{\sum x^2}{N} - \mu^2}$$

Where the symbols have same meaning as described in case of Population Variance.

**Illustration 44.** The following table shows the results of purity test on some compound. Find Variance and S.d.

Observed percent of impurity				
0.04	0.14	0.17	0.19	0.22
0.06	0.14	0.17	0.21	0.24
0.12	0.15	0.18	0.21	0.25

**Solution:**

Detailed computations are as follows:

**Calculations for Variance and S.d**

Sr No	Observation (x)	Deviation from mean (x - μ) = x - 0.166	(x - μ) <sup>2</sup>	x <sup>2</sup>
1	0.04	-0.126	0.016	0.0016
2	0.06	-0.106	0.011	0.0036
3	0.12	-0.046	0.002	0.0144
4	0.14	-0.026	0.001	0.0196
5	0.14	-0.026	0.001	0.0196
6	0.15	-0.016	0.000	0.0225
7	0.17	0.004	0.000	0.0289
8	0.17	0.004	0.000	0.0289
9	0.18	0.014	0.000	0.0324
10	0.19	0.024	0.001	0.0361
11	0.21	0.044	0.002	0.0441
12	0.21	0.044	0.002	0.0441
13	0.22	0.054	0.003	0.0484
14	0.24	0.074	0.005	0.0576
15	0.25	0.084	0.007	0.0625
Total	2.49	—	0.051	0.4643

$$\text{Mean } (\mu) = \frac{\sum x}{N} = \frac{2.49}{15} = 0.166$$

$$\text{Variance } (\sigma^2) = \frac{\sum (x - \mu)^2}{N} = \frac{0.051}{15} = 0.0034 \text{ percent squared}$$

OR

$$\text{Variance } (\sigma^2) = \frac{\sum x^2}{N} - \mu^2 = \frac{0.4643}{15} - (0.166)^2 = 0.0034 \text{ percent squared}$$

$$\text{S.d } (\sigma) = \sqrt{\sigma^2} = \sqrt{0.0034} = 0.058\%$$

Standard deviation enables us to determine, where the values of a frequency distribution are located in relation to the mean. We can measure the percentage of items that fall within specific ranges under a systematic bell shaped curve like the one below from where we can say that:

1. About 68 % of the values in the population will fall within plus and minus 1 standard deviation from the mean.
2. About 95 % of the values in the population will fall within plus and minus 2 standard deviation from the mean.
3. About 99 % of the values in the population will fall within plus and minus 3 standard deviation from the mean.

The standard deviation is also useful in describing how far individual items in a distribution disperse from the mean of the distribution. A measure called *Standard Score* gives us the number of standard deviations a particular observation lies below or above the mean.

Standard score computed from population data as =  $\frac{x - \mu}{\sigma}$

Suppose we observe a compound that is 0.108 percent impure. Since in the above Illustration 6 population has a mean of 0.166 and a standard deviation of 0.058, an observation of 0.108 would have a standard score of -1 as follows:

$$\text{Standard score} = \frac{x - \mu}{\sigma} = \frac{0.108 - 0.166}{0.058} = -1$$

For **Grouped data** variance and standard deviation are formulated as follows:

$$\sigma^2 = \frac{\sum f(x - \mu)^2}{\sum f} = \frac{\sum fx^2}{\sum f} - \mu^2$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum f(x - \mu)^2}{\sum f}} = \sqrt{\frac{\sum fx^2}{\sum f} - \mu^2}$$

Where:

$\sigma^2$  = population variance;

$\sigma$  = population standard deviation;

$x$  = Class mark of each class;

$\mu$  = population mean;

$f$  = frequency of each class

**Illustration 45.** From the following frequency distribution. Calculate variance and S.d

Annual Salary (₹ '000)	No of people
700-799	4
800-899	7
900-999	8
1000-1099	10
1100-1199	12
1200-1299	17
1300-1399	13
1400-1499	10
1500-1599	9
1600-1699	7
1700-1799	2
1800-1899	1

With such type of grouped data we can use the following formulae to calculate the variance and standard deviation:

$$\sigma^2 = \frac{\sum f(x - \mu)^2}{\sum f} = \frac{\sum fx^2}{\sum f} - \mu^2$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum f(x - \mu)^2}{\sum f}} = \sqrt{\frac{\sum fx^2}{\sum f} - \mu^2}$$

Where symbols have their usual meaning mentioned before

**Solution:**

**Calculations for Mean, Variance and S.d**

Class interval of Annual Salary	Class Mark (x)	Frequency (f)	fx	$x - \mu = x - 1249.5$	$(x - \mu)^2$	$f(x - \mu)^2$
700-799	749.5	4	2998	-500	250000	1000000
800-899	849.5	7	5946.5	-400	160000	1120000
900-999	949.5	8	7596	-300	90000	720000
1000-1099	1049.5	10	10495	-200	40000	400000
1100-1199	1149.5	12	13794	-100	10000	120000
1200-1299	1249.5	17	21241.5	0	0	0
1300-1399	1349.5	13	17543.5	100	10000	130000
1400-1499	1449.5	10	14495	200	40000	400000
1500-1599	1549.5	9	13945.5	300	90000	810000
1600-1699	1649.5	7	11546.5	400	160000	1120000
1700-1799	1749.5	2	3499	500	250000	500000
1800-1899	1849.5	1	1849.5	600	360000	360000
Total	—	100	124950	—	—	6680000

$$\mu = \frac{\sum (fx)}{\sum f} = \frac{124950}{100} = 1249.5$$

$$\sigma^2 = \frac{\sum f(x - \mu)^2}{\sum f} = \frac{6680000}{100} = 66800 \text{ Sq. ₹ ('000)}$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{66800} = ₹ 258.5 \text{ ('000)}$$

Standard deviation cannot be computed from open ended distributions. It considers every observation but extreme values in the data set distort the value of the standard deviation, although to a lesser extent than do the range.

**Illustration 46.** Calculate the population variance for the following set of grouped data:

Class	0-199	200-399	400-599	600-799	800-999
Frequency	8	13	20	12	7

**Solution:**

We know that population variance is computed as  $\sigma^2 = \frac{\sum f(x-\mu)^2}{N} = \frac{\sum fx^2}{N} - \mu^2$

**Calculations for Variance**

Class	Class Mark (x)	Frequency (f)	fx	$x - \mu$ = $x - 489.5$	$(x - \mu)^2$	$f(x - \mu)^2$
0-199	99.5	8	796	-390	152100	1216800
200-399	299.5	13	3893.5	-190	36100	469300
400-599	499.5	20	9990	10	100	2000
600-799	699.5	12	8394	210	44100	529200
800-999	899.5	7	6296.5	410	168100	1176700
Total	–	60	29370	–	–	3394000

$$\text{Mean} = \mu = \frac{\sum fx}{\sum f} = \frac{29370}{60} = 489.5$$

$$\text{Variance} = \sigma^2 = \frac{\sum f(x-\mu)^2}{\sum f} = \frac{3394000}{60} = 56566.67$$

Generally we calculate variance and standard deviation for ungrouped & grouped data through various formulae as given below.

**1. Ungrouped data:** (a) Direct Method; (b) Shortcut Method; (c) Step deviation method

Direct method:

$$(a) \sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$$

$$(b) \sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

Shortcut Method

$$(a) \sigma = \sqrt{\frac{\sum dx^2}{n} - \left(\frac{\sum dx}{n}\right)^2}, \text{ where } dx = x - A, A \text{ is the assumed mean}$$

Step Deviation Method

$$(b) \sigma = \sqrt{\frac{\sum dx'^2}{n} - \left(\frac{\sum dx'}{n}\right)^2} \times h$$

where  $dx' = \frac{x-A}{h}$ ,  $A$  is the assumed mean,  $h$  is the common factor

**2. Grouped data:** (a) Direct Method; (b) Shortcut Method; (c) Step deviation method

Direct method:

$$(a) \sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}}$$

$$(b) \sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

Shortcut Method

$$(a) \sigma = \sqrt{\frac{\sum fdx^2}{\sum f} - \left(\frac{\sum fdx}{\sum f}\right)^2}, \text{ where } dx = x - A, A \text{ is the assumed mean}$$

Step Deviation Method

$$(b) \sigma = \sqrt{\frac{\sum fdx'^2}{\sum f} - \left(\frac{\sum fdx'}{\sum f}\right)^2} \times h$$

Where  $dx' = \frac{x - A}{h}$ , A is the assumed mean, h is the common factor (Generally it is the Common Width of the class intervals).

**3. Standard Deviation of combined Series:**

If two groups contain  $n_1$  and  $n_2$  observations with mean  $\bar{X}_1$  &  $\bar{X}_2$  and standard deviations  $\sigma_1$  &  $\sigma_2$  respectively, the S.D ( $\sigma_{12}$ ) of the combined group is given by

$$\sigma_{12} = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$$

Where

$$d_1 = \bar{X}_{12} - \bar{X}_1$$

$$d_2 = \bar{X}_{12} - \bar{X}_2$$

$$\bar{X}_{12} = \text{Combined mean} = \frac{n_1 \cdot \bar{X}_1 + n_2 \cdot \bar{X}_2}{n_1 + n_2}$$

**Illustration 47.** Calculate the standard deviation of the following distribution:

Age	No of people
20-25	170
25-30	110
30-35	80
35-40	45
40-45	40
45-50	35

**Solution:**

Let the assumed mean is  $A = 32.5$

We have, Standard Deviation =  $\sigma = \sqrt{\frac{\sum fdx^2}{\sum f} - \left(\frac{\sum fdx}{\sum f}\right)^2}$  according to Shortcut Method.

The following table shows how to apply the above formula to find the standard deviation from the given grouped data.

**Calculations for S.d**

Class	Class Mark (x)	Frequency (f)	$d = x - A$ $= x - 32.5$	$fd$	$fd^2$
20-25	22.5	170	-10	-1700	17000
25-30	27.5	110	-5	-550	2750
30-35	32.5	80	0	0	0
35-40	37.5	45	5	225	1125
40-45	42.5	40	10	400	4000
45-50	47.5	35	15	525	7875
Total	–	480	–	-1100	32750

$$\sigma = \sqrt{\frac{32750}{480} - \left(\frac{-1100}{480}\right)^2} = \sqrt{68.23 - 5.25} = 7.936$$

Another method of calculating standard deviation is called Step deviation method by which

$$\sigma = \sqrt{\frac{\sum fd'^2}{N} - \left(\frac{\sum fd'}{N}\right)^2} \times h$$

Where  $d' = \frac{x - A}{h}$ ,

$h$  is a common factor,  $A$  is the assumed average and  $x$  is the mid value of the class interval

**Illustration 48.** Calculate the mean and standard deviation from the following data:

Value	Frequency
90-99	2
80-89	12
70-79	22
60-69	20
50-59	14
40-49	4
30-39	1

**Solution:**

**Calculations for Mean and S.d**

Class Interval with Class Limits	Class Interval with Class Boundaries	Mid value (x)	$d' = \frac{x - 64.5}{10}$	f	fd'	f d' <sup>2</sup>
90-99	89.5 - 99.5	94.5	3	2	6	18
80-89	79.5 - 89.5	84.5	2	12	24	48
70-79	69.5 - 79.5	74.5	1	22	22	22
60-69	59.5 - 69.5	64.5	0	20	0	0
50-59	49.5 - 59.5	54.5	-1	14	-14	14
40-49	39.5 - 49.5	44.5	-2	4	-8	16
30-39	29.5 - 39.5	34.5	-3	1	-3	9
Total	–	–	–	75	27	127

In the above computation

$$A = 64.5, h = 10$$

$$\sigma = \sqrt{\frac{\sum fd'^2}{\sum f} - \left(\frac{\sum fd'}{\sum f}\right)^2} \times h = \sqrt{\frac{127}{75} - \left(\frac{27}{75}\right)^2} \times 10 = \sqrt{1.6933 - 0.1296} \times 10 = 12.505$$

$$\bar{X} = A + \frac{\sum fd'}{\sum f} \times h = 64.5 + \frac{27}{75} \times 10 = 68.1$$

**Illustration 49.** Calculate standard deviation from the following set of observations

8	9	15	23	5	11	19	8	10	12
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**Solution:**

These are ungrouped data. For these sort of data standard deviation is computed by the following formula –

Direct method:

$$\sigma = \sqrt{\frac{\sum (X - \bar{X})^2}{n}}, \text{ where } \bar{X} \text{ is the arithmetic mean, } n = \text{number of items} \dots \dots \dots \text{(A)}$$

$$\text{Or, } \sigma = \sqrt{\frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2} \dots \dots \dots \text{(B)}$$

The detail computation as per formula (A) and (B) are given in the following tables

**Calculations for S.d using Equation (A)**

Sl. No.	Values (X)	Deviation $x = X - \bar{X} = X - 12$	Deviation Squared $x^2$
1	8	-4	16
2	9	-3	9
3	15	3	9
4	23	11	121
5	5	-7	49
6	11	-1	1
7	19	7	49
8	8	-4	16
9	10	-2	4
10	12	0	0
Total	120	–	274

$$\bar{X} = \Sigma X/n = 12$$

$$\text{So } \sigma = \sqrt{\frac{274}{10}} = 5.23.$$

**Alternative Method-1**

**Calculations for S.d using Equation (B)**

Sl. No.	Values (X)	Square of Values (X <sup>2</sup> )
1	8	64
2	9	81
3	15	225
4	23	529
5	5	25
6	11	121
7	19	361
8	8	64
9	10	100
10	12	144
Total	120	1714

$$\text{So } \sigma = \sqrt{\frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2} = \sqrt{\frac{1714}{10} - \left(\frac{120}{10}\right)^2} = 5.23$$

**Alternative Method-2**

The s.d of above ungrouped data can also be found by short cut method by the following formula:

$$\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

where  $d = X - A$  and  $A$  is assumed mean = 10, in this case.

The detailed computation is given in the following table

**Calculation for s.d**

Values (X)	Deviation $d = X - A = X - 10$	$d^2$
8	-2	4
9	-1	1
15	5	25
23	13	169
5	-5	25
11	1	1
19	9	81
8	-2	4
10	0	0
12	2	4
Total	20	314

$$\text{So } \sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2} = \sqrt{\frac{314}{10} - \left(\frac{20}{10}\right)^2} = 5.23$$

**Illustration 50.** The analysis of the results of a budget survey of 150 families gave an average monthly expenditure of ₹120 on food items with a standard deviation of ₹15. After the analysis was completed it was noted that figure recorded for one household was wrongly taken as ₹15 instead of ₹105. Determine the correct value of the average expenditure and its standard deviation.

**Solution:** We are given  $n = 150$ ,  $\bar{X} = 120$  and  $\sigma = 15$

Wrong value used =15 and correct value =105

$$\text{Now } \bar{X} = \frac{\sum X}{n} \text{ or } 120 = \frac{\sum X}{150}$$

So that  $\sum X = 120 \times 150 = 18000$

Corrected  $\sum X = 18000 - 15 + 105 = 18090$

$$\text{Moreover } \sigma^2 = \frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2$$

$$\text{Or, } (15)^2 = \frac{\sum X^2}{150} - (120)^2$$

$$\text{So } \Sigma X^2 = 2193750$$

$$\text{Therefore Corrected } \Sigma X^2 = 2193750 - 15^2 + 105^2 = 2204550$$

$$\text{So Corrected Mean} = \frac{18090}{150} = ₹ 120.6$$

$$\text{Corrected Variance.} = \sigma^2 = \frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2 = \frac{2204550}{150} - (120.6)^2 = 152.64$$

$$\text{So Corrected S.d} = \sigma = \sqrt{152.64} = ₹ 12.35$$

**Illustration 51.** For a group containing 100 observations, the arithmetic mean and standard deviations are 8 and  $\sqrt{10.5}$ . For 50 observations selected from these 100 observations, the mean and the standard deviation are 10 and 2 respectively. Find the arithmetic mean and standard deviation of the other half.

**Solution:**

Suppose the 100 observations have been split up into two groups – Group I containing 50 observations with mean  $\bar{X}_1 = 10$  and standard deviation  $\sigma_1 = 2$  and Group II containing the remaining 50 observations. The combined group of 100 observations has mean  $\bar{X}_{12} = 8$  and s.d.  $= \sigma_{12} = \sqrt{10.5}$ . It is required to find the mean  $\bar{X}_2$  and s.d. ( $\sigma_2$ ) of group II.

The values are shown in the following table:

Characteristics	Groups		Combined Group
	Group I	Group II	
No of observations	$n_1 = 50$	$n_2 = 50$	100
Mean	$\bar{X}_1 = 10$	$\bar{X}_2 = ?$	$\bar{X}_{12} = 8$
S.D.	$\sigma_1 = 2$	$\sigma_2 = ?$	$\sigma_{12} = \sqrt{10.5}$

$$\text{The mean of combined group} = \bar{X}_{12} = \frac{n_1 \bar{X}_1 + n_2 \bar{X}_2}{n_1 + n_2} \text{ or, } \frac{50 \times 10 + 50 \times \bar{X}_2}{50 + 50} = 8 \text{ (Given)}$$

$$\text{From above we have } \bar{X}_2 = 6$$

Again the standard deviation of the combined group is

$$\sigma_{12} = \sqrt{\frac{(n_1 \sigma_1^2 + n_2 \sigma_2^2) + (n_1 d_1^2 + n_2 d_2^2)}{n_1 + n_2}}$$

$$\text{Where } d_1 = \bar{X}_1 - \bar{X}_{12} \text{ \& } d_2 = \bar{X}_2 - \bar{X}_{12}$$

$$d_1 = 10 - 8 = 2$$

$$d_2 = 6 - 8 = -2$$

We can write from the s.d. formula of the combined group above —

$$(n_1 + n_2) \sigma_{12}^2 = (n_1 \sigma_1^2 + n_2 \sigma_2^2) + (n_1 d_1^2 + n_2 d_2^2)$$

$$\text{Or, } (50 + 50) 10.5 = (50 \times 2^2 + 50 \times \sigma_2^2) + [50 \times 2^2 + 50 \times (-2)^2]$$

$$\text{From here we have } \sigma_2 = 3$$

**Illustration 52.**

If S.D of 1st 'n' natural nos be 2, Find 'n'.

**Solution:**  $\frac{n^2 - 1}{12} = 4$   
 or  $n^2 - 1 = 48$       or,  $n^2 = 49$       or  $n = 7$ .

**Illustration 53.**

If x and y related by  $2x + 3y = 40$  and S.D of x is 3, find S.D of y.

**Solution:**  $3y = 40 - 2x$   
 Or  $y = \frac{40}{3} - \frac{2x}{3}$   
 S.D of  $y = \frac{2}{3}$  S.D of  $x = \frac{2}{3} \times 3 = 2$

**Illustration 54.**

If S.D of x is 2, then find S.D of  $(3x - 2)$ .

**Solution:** S.d of  $(3x - 2) = 3 \times$  S.D of  $x = 3 \times 2 = 6$ .

**Illustration 55.**

If A.M =  $\frac{8+4}{2}$  find variance.

**Solution:** S.D =  $\frac{8-4}{2} = 2$   
 Var = S.D<sup>2</sup> = 4

**Illustration 56.**

If  $\text{Var}(x) = 4$ . Find variance of  $(2x - 8)$

**Solution:** S.D of  $x = 2$   
 S.D of  $(2x - 8) = 2 \times$  S.D of  $x = 2 \times 2 = 4$   
 Var  $(2x - 8) = 4^2 = 16$ .

**Illustration 57.**

The mean of 2 nos. is 20 and their S.D 5, Find two nos.

**Solution:**  $\frac{a+b}{2} = 20$       or  $a + b = 40$ .      -----(1)  
 $\frac{a-b}{2} = 5$       or,  $a - b = 10$       -----(2)  
 Solving 1 & 2  
 $a = 25, b = 15$

**Illustration 58.** Find Range and Coefficient of Range for the following data

X	5	10	15	20	25	30	35	40
f	4	7	21	47	53	24	12	6

**Solution:**

Going through the values of  $X$  we get smallest value  $S = 5$  and largest value  $L = 40$

Range =  $L - S = 40 - 5 = 35$ . It is an absolute measure.

Coefficient of Range =  $\frac{L - S}{L + S} = \frac{40 - 5}{40 + 5} = 0.77$ . It is a relative measure.

**Illustration 59.** Calculate Range and Coefficient of Range for following data

X	20-30	30-40	40-50	50-60	60-70	70-80
f	4	9	16	21	13	6

**Solution:**

Here  $L = 80$  and  $S = 20$

$\therefore$  Range =  $L - S = 80 - 20 = 60$

And Coefficient of Range =  $\frac{L - S}{L + S} = \frac{80 - 20}{80 + 20} = 0.6$

**Illustration 60.** Compute Inter quartile Range, Coefficient of Quartile Deviation and Percentile Range for the following data.

X	4	8	12	16	20	24	28	32
f	4	9	17	40	53	37	24	16

**Solution:**

Computation details are

	X	f	C.f.	
	4	4	4	
	8	9	13	
$P_{10}$ →	12	17	30	← 20.1
$Q_1$ →	16	40	70	← 50.25
	20	53	123	
$Q_3$ →	24	37	160	← 150.75
$P_{90}$ →	28	24	184	← 180.9
	32	16	200	

Ranks of  $Q_1 = (200 + 1)/4 = 50.25$  i.e 50.25th term is  $Q_1$ . From the above Table we find that 31st to 70th all the values are 16. Thus 50.25th value should be 16. Hence  $Q_1 = 16$ .

Ranks of  $Q_3 = 3(200 + 1)/4 = 150.75$  i.e 150.75th term is  $Q_3$ . Using the similar concept as above  $Q_3 = 24$ .

$$\text{Inter Quartile Range} = Q_3 - Q_1 = 24 - 16 = 8$$

$$\text{Coefficient of Q.D} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{24 - 16}{24 + 16} = \frac{8}{40} = 0.2$$

To Compute Percentile Range:

$$\text{Rank of } P_{90} = 90 \times (200+1)/100 = 180.9 \text{ i.e } 180.9\text{th term is } P_{90}$$

$$\text{So } P_{90} = 28$$

$$\text{Rank of } P_{10} = 10 \times (200+1)/100 = 20.1 \text{ i.e } 20.1\text{th term is } P_{10}$$

$$\text{So } P_{10} = 12$$

$$\text{Percentile Range} = P_{90} - P_{10} = 28 - 12 = 16$$

**Illustration 61.** Compute M.D. and Coefficient of M.D. from mean and median for the following series.

X	3	7	12	14	15	18	22
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**Solution:**

Detail computations of mean and median (14 as found from data) are:

**Calculations for M.D from Mean & Median**

Sl. No.	X	X-Mean	X-Median
1	3	10	11
2	7	6	7
3	12	1	2
4	14	1	0
5	15	2	1
6	18	5	4
7	22	9	8
Total	91	34	33

$$\text{Mean} = \Sigma X/n = 91/7 = 13$$

$$\text{Mean Deviation from mean} = \frac{\Sigma |(X - \bar{X})|}{n} = \frac{34}{7} = 4.86$$

$$\text{Mean Deviation from median} = \frac{\Sigma |(X - M_d)|}{n} = \frac{33}{7} = 4.71$$

$$\text{Coefficient of Mean Deviation from Mean} = \frac{\text{Mean Deviation from Mean}}{\text{Mean}} = \frac{4.86}{13} = 0.37$$

$$\text{Coefficient of Mean Deviation from Median} = \frac{\text{Mean Deviation from median}}{\text{Median}} = \frac{4.71}{14} = 0.34$$

**Illustration 62.** Compute M.D. from Mean and Median for the given series:

X	5	10	15	20	25	30
f	3	4	8	12	7	2

**Solution:**

Detail Computation of M. D from mean and median are:

**Calculations for M.D about Mean and Median**

X	f	C.f.	Xf	X-Mean	f X-Mean	X-Median	f X-Median
5	3	3	15	13.06	39.18	15	45
10	4	7	40	8.06	32.24	10	40
15	8	15	120	3.06	24.48	5	40
20	12	27	240	1.94	23.28	0	0
25	7	34	175	6.94	48.58	5	35
30	2	36	60	11.94	23.88	10	20
Total	36	–	650	–	191.64	–	180.000

$$\text{Mean} = \frac{\sum Xf}{\sum f} = \frac{650}{36} = 18.06$$

$$\text{Mean Deviation from mean} = \frac{\sum f|(X - \bar{X})|}{\sum f} = \frac{191.64}{36} = 5.323$$

$\sum f = 36$  Thus Median class is  $36/2 = 18^{\text{th}}$  c.f class. So Median is 20

$$\text{Mean Deviation from median} = \frac{\sum f|(x - M_d)|}{\sum f} = \frac{180}{36} = 5$$

**Illustration 63.**

If Q.D. = 60, Find M.D.

**Solution:**

$$\text{M.D.} \frac{6}{5} = \text{Q. D} = \frac{6}{5} \times 60 = 72$$

**Illustration 64.**

Find co-efficient of mean deviation about median for the 1st 5 natural nos.

**Solution:**

$$\begin{aligned}
 \text{M.D about median} &= \frac{\sum |x_i - M_e|}{n} \\
 &= \frac{|1 - 3| + |2 - 3| + |3 - 3| + |4 - 3| + |5 - 3|}{5} \\
 &= \frac{2+1+0+1+2}{5} \\
 &= \frac{6}{5}
 \end{aligned}$$

Coefficient of M.D about median

$$\begin{aligned}
 &= \frac{\text{M.D about median}}{\text{median}} \times 100 \\
 &= \frac{6}{5 \times 3} \times 100 = 40\% \quad (\text{Me} = 3)
 \end{aligned}$$

**Illustration 65.**Which is the value of M.D, when  $Q_3 = 40$  &  $Q_1 = 15$ .**Solution:**

$$6 \text{ Q.D} = 5 \text{ M.D}$$

$$\text{Or } 6 \frac{Q_3 - Q_1}{2} = 5 \text{ M.D}$$

$$\text{Or } 6 \times \frac{40 - 15}{2} = 5 \text{ M.D}$$

$$\text{Or } \frac{3(25)}{5} = \text{M.D} = 15$$

# Co-efficient of Variation

## 5.4

### Coefficient of Variation (Coefficient of variability):

The relative measure of dispersion based upon standard deviation is called *Coefficient of Variation*. The formula is

$$\text{Coefficient of variation (C.V)} = \frac{\text{Standard Deviation}}{\text{Mean}} \times 100$$

- ⊙ Coefficient of variation being a pure number is independent of the units of measurement and thus is suitable for comparing the variability of two or more series of data.
- ⊙ Higher the value of CV more will be the variability of the data series under consideration.

**Illustration 66.** From some financial statistics, it is found that the monthly average electricity charges were ₹2400 and S.D. ₹120. The monthly average direct wages were ₹42000 and S.D. ₹1200. Which one of electricity charges and direct wages is more variable?

#### Solution:

$$\text{Coefficient of variation for Electricity charges} = \frac{\text{Standard Deviation}}{\text{Mean}} \times 100 = \frac{120}{2400} \times 100 = 5\%$$

$$\text{Coefficient of variation for Direct wages} = \frac{\text{Standard Deviation}}{\text{Mean}} \times 100 = \frac{1200}{42000} \times 100 = 2.9\%$$

So electricity charges is more variable

**Illustration 67.** If mean of a series is 10 and its coefficient of variation is 40%, then find S.D.

#### Solution:

$$\text{C.V} = \frac{\text{S.D.}}{\text{mean}} \times 100$$

$$\text{or } 40 = \frac{\text{S.D.}}{\text{mean}} \times 100$$

$$\text{or } \frac{40 \times 10}{100} = \text{S.D} = 4.$$

### Properties of Standard Deviations:

1. S.D. is independent of change of origin.

For example S.D. of 3, 4, 10, 19 will be same as that of 5, 6, 12, 21. Here 2 has been added to each term of the 1st Series to get the 2nd Series, but S.D. remains same.

2. S.D. is not independent of change of scale.

For example S.D. of 2, 4, 7, 9 is half of the S.D of 4, 8, 14, 18 as terms are half of the latter series.

3. Sum of squares taken from the Arithmetic Mean is always minimum i.e.  $\Sigma(X - \bar{X})^2$  is always least.

4. S.D. of first n natural number is  $\sqrt{\frac{1}{12}(n^2 - 1)}$

5. For a given set of observation *Standard Deviation*  $\geq$  *Mean Deviation*

**Relation between various measures of Dispersion:**

Q.D = 2/3 (S.D) and M.D = 4/5 (S.D)

In case of QD, MD and SD, the relation is

- (1)  $\bar{X} \pm$  QD covers 50% of the total terms.
- (2)  $\bar{X} \pm$  MD covers 57.51% of the total terms.
- (3)  $\bar{X} \pm$  SD covers 68.27% of the total terms.
- (4)  $\bar{X} \pm 2$ SD covers 95.45% of the total terms.
- (5)  $\bar{X} \pm 3$ SD covers 99.73% of the total terms.

**Conclusion:** If we discuss about the various measures of dispersion we come to Range first. But Range is not a stable measure and has many drawbacks such as fluctuations of sampling. Thus can't be taken as a good measure of dispersion.

**Quartile Deviation:** It is certainly better than Range. But here all the terms are not taken into account. It also suffers from sampling instability. It can be effectively used when class intervals are open; and to calculate M.D and S.D. these limits have to be assumed.

**Mean Deviation:** It is not capable of further Algebraic Treatment, although it takes into account all the terms but still, if the extreme values are big it will distort the result. Moreover, it ignores plus and minus signs; therefore can't be called a good measure of dispersion.

**Standard Deviation:** It is the most important measure of dispersion and is widely used. In this, the drawback of ignoring the algebraic sign as in Mean Deviation is overcome by taking the square of deviations thereby making all the deviations positive.

**Formula Recapitulation:**

Important formulae on dispersion are:

- ⊙ Range = L – S, Where L is the largest term, S is the smallest term;
- ⊙ Coefficient of Range =  $\frac{L - S}{L + S}$ ;
- ⊙ Interquartile range =  $Q_3 - Q_1$  where  $Q_1$  is lower quartile and  $Q_3$  is the upper quartile;
- ⊙ Semi Interquartile range or quartile deviation =  $\frac{Q_3 - Q_1}{2}$ ;
- ⊙ Coefficient of quartile deviation =  $\frac{Q_3 - Q_1}{Q_3 + Q_1}$ ;

- ⊙ Percentile Range =  $P_{90} - P_{10}$ ;
- ⊙ Decile Range =  $D_9 - D_1$ ;
- ⊙ Mean deviation or average deviation MD =  $\frac{\sum f|D|}{N}$ , where  $|D| = |X - A|$ ,  $N = \sum f$ ,  $A = \text{Mean, median or mode}$ ;
- ⊙ Coefficient of mean deviation =  $\frac{MD}{A}$
- ⊙ Standard deviation for discrete data =  $\sigma = \sqrt{\frac{\sum (X - \bar{X})^2}{N}}$ , where  $\bar{X}$  is the mean and  $N$  total no of items;
- ⊙ Standard deviation for discrete data from assumed mean =  $\sigma = \sqrt{\frac{\sum dx^2}{N} - \left(\frac{\sum dx}{N}\right)^2}$  where  $dx = X - A$ ,  $A$  is assumed mean = any number
- ⊙ Standard deviation for grouped data =  $\sigma = \sqrt{\frac{\sum f dx^2}{N} - \left(\frac{\sum f dx}{N}\right)^2}$  where  $dx = X - A$ ,  $A$  is assumed mean = any number or actual mean and  $N = \sum f$
- ⊙ Standard deviation of combined group – Group I and Group II

$$\sigma_{12} = \sqrt{\frac{(n_1\sigma_1^2 + n_2\sigma_2^2) + (n_1d_1^2 + n_2d_2^2)}{n_1 + n_2}}$$

Where

$\sigma_1 = \text{S.d of Group I}$ ,  $\sigma_2 = \text{S. d of Group II}$ ,  $n_1 = \text{no of items in Group I}$ ,  $n_2 = \text{no of items in Group II}$ ,  $d_1$  &  $d_2$  are deviations of group means from combined group mean

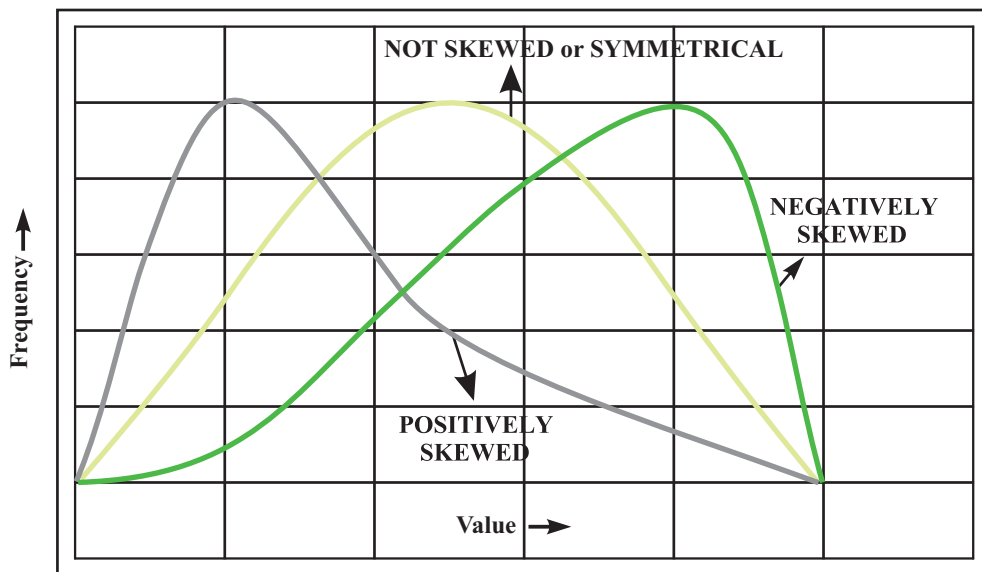
- ⊙ Coefficient of Variation =  $\frac{\text{S.D}}{\text{Mean}} \times 100$

# Karl Pearson and Bowley's Co-efficient of Skewness

## 5.5

### Skewness:

Skewness refers to the asymmetry or lack of symmetry in the shape of a frequency distribution curve.



**Figure-1:** Showing typical Frequency Distribution curves with different types of skewness

So skewness is the attribute of a frequency distribution that extends further on one side of the class with the highest frequency than on the other.

Let us take the following frequency distribution (Table 1) of three groups –Group I, Group II and Group III

For each group  $\Sigma f = 250$  and size pattern same as shown.

For each group we find Average as  $\bar{X} = \frac{\Sigma xf}{\Sigma f}$ , Median as  $M_d = l_1 + \frac{\frac{N}{2} - C}{f} \times h$  and Mode as

$$M_o = l_1 + \frac{f_1 - f_0}{2 \times f_1 - f_0 - f_2} \times h$$

Where

$x$  = mid values of different classes,  $f$  = frequency,  $N = \sum f$ ,  $l_1$  = lower boundary of Median or Modal class,  $C$  = cumulative frequency of class just preceding median class,  $f$  in median formula = frequency of median class,  $f_1$  = frequency of modal class,  $f_0$  = frequency of the class just preceding modal class,  $f_2$  = frequency of the class just succeeding modal class,  $h$  = width of median and modal class.

Size	Mid value	Group I Frequency-I	Group II Frequency-II	Group III Frequency-III
0 – 5	2.5	10	10	10
5 – 10	7.5	90	30	20
10 – 15	12.5	50	50	30
15 – 20	17.5	40	70	40
20 – 25	22.5	30	50	50
25 – 30	27.5	20	30	90
30 – 35	32.5	10	10	10
Total	–	250	250	250

**Table -I**

From the above data and formula given for Mean Median and Mode we have the following result, for different groups.

	Group I	Group II	Group III
Mean	14.3	17.5	20.7
Median	12.5	17.5	26.25
Mode	8.33	17.5	26.67

From the above table we can say

For group I: Mean > Median > Mode

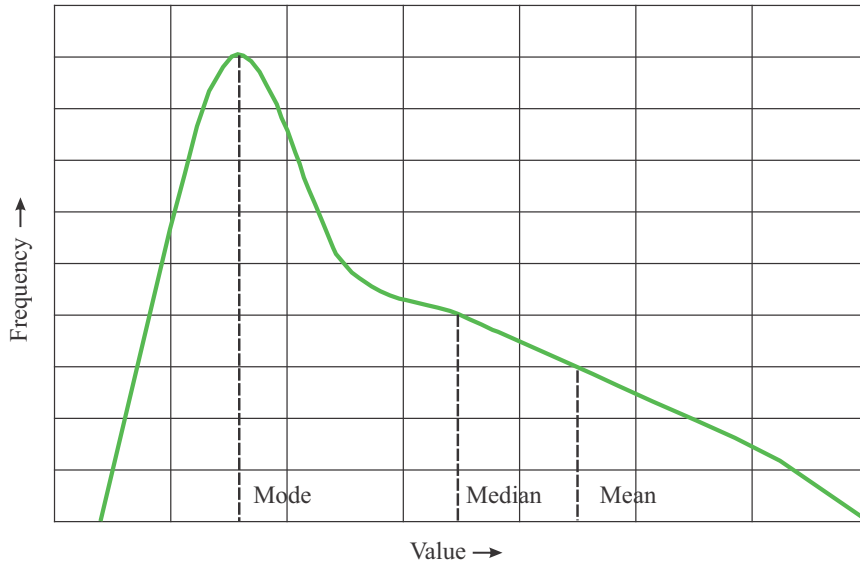
For group II: Mean = Median = Mode

For group III: Mean < Median < Mode

For these three groups if we plot the data in a graph paper then we will observe the following:

**For Group I**

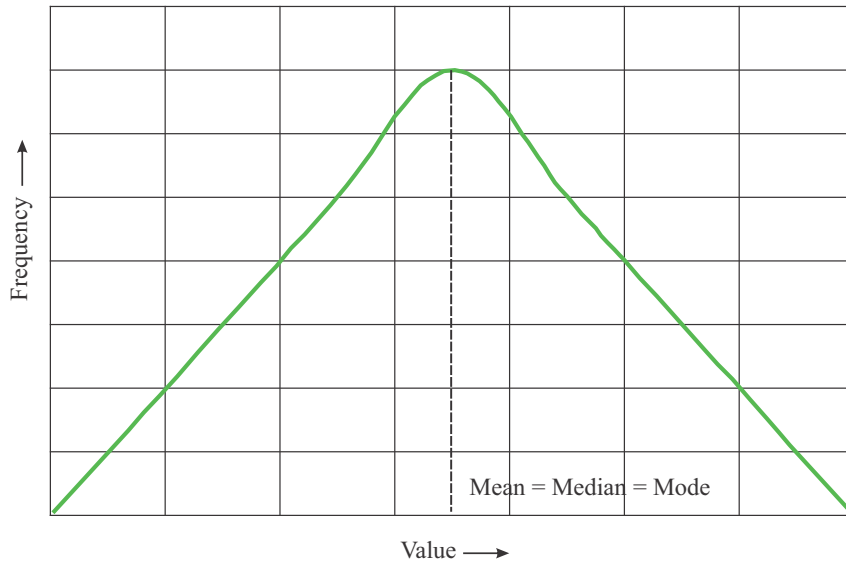
- Refer figure 2— a positively skewed curve— a positively skewed curve occurs when a frequency distribution has  $\text{Mean} > \text{Median} > \text{Mode}$



**Figure-2:** Showing a typical Positivity Skewed Frequency distribution curve

**For Group II**

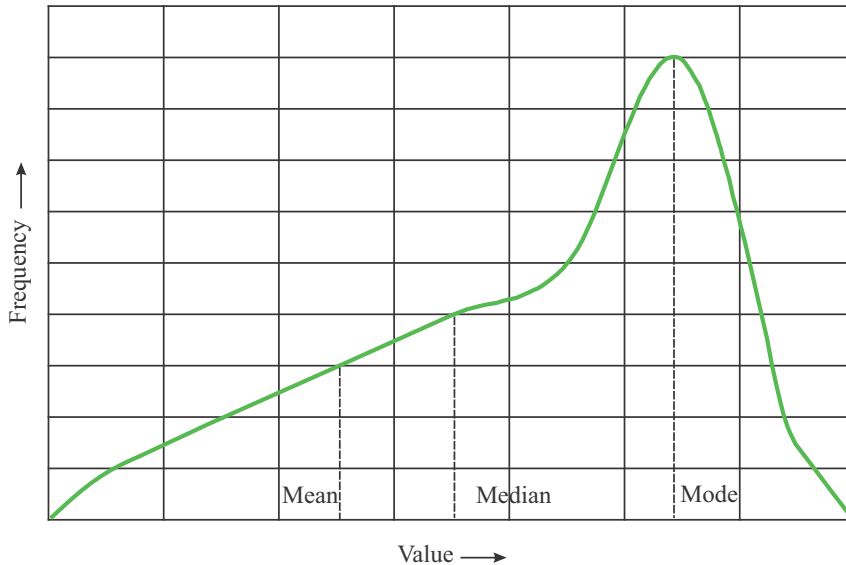
- Refer figure 3— a symmetrical curve— a symmetrical curve occurs when a frequency distribution has  $\text{Mean} = \text{Median} = \text{Mode}$



**Figure-3:** Showing a typical Symmetrical Frequency distribution curve

**For Group III**

- Refer figure 4—a negatively skewed curve—a negatively skewed curve occurs when a frequency distribution has  $\text{Mean} < \text{Median} < \text{Mode}$



**Figure-4:** Showing a typical Negatively Skewed Frequency distribution curve

From the above discussion it can be concluded that

In case of Symmetrical Series, the mean, median and mode coincide. i.e.,  $\text{Mean} = \text{Median} = \text{Mode}$

In case of Positive Skewed: If the tail is towards right, then it is called positive skewness. It means  $\text{Mode} < \text{Median} < \text{Mean}$

In case of Negative Skewed: When the tail of a distribution is towards left, then the skewness is negative i.e.  $\text{Mode} > \text{Median} > \text{Mean}$

It should be noted that in both '+ve and '-ve skewed distributions that median lies in between the mode and the mean.

Other characteristics of Positively Skewed (Group -I), Symmetrical (Group-II) and Negatively Skewed (Group-III) distributions are

For Group -I :  $Q_3 - M_d > M_d - Q_1$

For Group -II :  $Q_3 - M_d = M_d - Q_1$

For Group -III :  $Q_3 - M_d < M_d - Q_1$

Where  $M_d$  is the median  $Q_3$  is the third quartile,  $Q_1$  is the first quartile

There are three Absolute measures of skewness:

Absolute skewness (SK) = Mean – Mode

Absolute skewness (SK) = Mean – Median

Absolute skewness (SK) = Median – Mode

There are four types of relative measures of skewness namely:

- (1) The Karl Pearson’s Coefficient of Skewness;
- (2) The Bowley’s Coefficient of Skewness;
- (3) The Kelly’s Coefficient of Skewness;
- (4) Measurement of Skewness based on Moments;

**The Karl Pearson’s Coefficient of Skewness:**

$$Sk = \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}}$$

If in a particular frequency distribution, it is difficult to determine precisely the mode, or the mode is ill defined then:

$$Sk = \frac{3(\text{Mean} - \text{Median})}{\text{Standard Deviation}}$$

Skewness obtained by both of these formulae may vary between ±3. But mostly it varies in the range ±1.

For our Groups the Skewness values as per this formula are given in the following table:

Group I	Group II	Group III
0.77184	0	-0.77184

Skewness value of Group I is positive and that of Group III is negative, as observed from graph. Also Grpah II being symmetrical shows zero skewness.

**Bowley’s Coefficient of Skewness:**

$$Sk = \frac{Q_3 + Q_1 - 2M_d}{Q_3 - Q_1}$$

where  $Q_3$  &  $Q_1$  are third and first quartile and  $M_d$  is the Median of the distribution

This is also called Quartile measure of skewness and it varies between +1 and -1.

For our Groups the Skewness values as per this formula are given in the following table:

	Group I	Group II	Group III
Q3	19.69	22.75	27.08
Md	12.50	17.50	22.50
Q1	7.92	12.25	15.31
Sk	0.22	0.00	-0.22

While obtaining the above results the following formula are used for determining standard deviation, third quartile and first quartile:

$$\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \begin{array}{|c|c|c|} \hline \text{Group I} & \text{Group II} & \text{Group III} \\ \hline 7.73 & 7.21 & 7.73 \\ \hline \end{array}$$

$$Q_3 = l_1 + \frac{\frac{3N}{4} - C}{f} \times h \quad \& \quad Q_1 = l_1 + \frac{\frac{N}{4} - C}{f} \times h$$

Where  $l$  = lower boundary of the respective quartile class

$C$  = cumulative frequency of the class just preceding the respective quartile class

$f$  = simple frequency of the respective quartile class

$h$  = width of the respective quartile class

In practice Karl Pearson's Coefficient of Skewness is usually used.

### The Kelly's Coefficient of Skewness:

$$Sk = \frac{P_{90} + P_{10} - 2P_{50}}{P_{90} - P_{10}}$$

OR

$$Sk = \frac{D_9 + D_1 - 2D_5}{D_9 - D_1}$$

Kelly's Coefficient of Skewness is also called percentile skewness or decile skewness. It is seldom used in practice. Its value lies between  $\pm 1$ .

**Illustration 68.** For a group of 10 items,  $\Sigma X = 452$ ,  $\Sigma X^2 = 24270$ , and Mode = 43.7; Find the Pearson's coefficient of Skewness

### Solution:

$$\text{Mean } (\bar{X}) = \frac{\Sigma X}{n} = \frac{452}{10} = 45.2$$

$$\text{S.D. } (\sigma) = \sqrt{\frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2} = \sqrt{\frac{24270}{10} - (45.2)^2} = 19.59$$

Pearson's coefficient of skewness:

$$Sk = \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}} = \frac{45.2 - 43.7}{19.59} = 0.08$$

### Illustration 69.

Consider the following distribution and comment on the correctness of the statements (i) and (ii) with reasons.

	Distribution A	Distribution B
Mean	100	90
Median	90	80
Standard Deviation	10	10

- (i) Distribution A has the same degree of the variation as distribution B  
 (ii) Both distributions have the same degree of skewness.

**Solution:**

$$(i) \text{ Coefficient of variation (for distribution A)} = \frac{\sigma_A}{\bar{X}_A} \times 100 = \frac{10}{100} \times 100 = 10\%$$

$$\text{Coefficient of variation (for distribution B)} = \frac{\sigma_B}{\bar{X}_B} \times 100 = \frac{10}{90} \times 100 = 11.11\%$$

Since C.V. of (B) > C.V. of (A), the distribution B is more variable than the distribution A. Thus the given statement that the distribution A has the same degree of variation as distribution B is false.

(ii) Karl Pearson's coefficient of skewness for distribution A and B is given by

$$\text{Sk (A)} = \frac{3(\text{Mean} - \text{Median})}{\sigma} = \frac{3(100 - 90)}{10} = 3$$

$$\text{Sk (B)} = \frac{3(\text{Mean} - \text{Median})}{\sigma} = \frac{3(90 - 80)}{10} = 3$$

Thus the given statement that both distributions have same degree of skewness is true.

**Illustration 70.** You are given below the following details relating to wages in respect of two factories from which it is concluded that the skewness and variability are the same in both the factories:

	Factory A	Factory B
Mean	₹50	₹45
Mode	₹45	₹50
Variance	Sq. ₹100	Sq. ₹100

Point out the mistake, if any, in the above statement

**Solution:**

$$\text{Standard deviation} = \sigma = \sqrt{\text{variance}} = \sqrt{100} = 10$$

$$\text{Coefficient of variation (for distribution A)} = \frac{\sigma_A}{\bar{X}_A} \times 100 = \frac{10}{50} \times 100 = 20\%$$

$$\text{Coefficient of variation (for distribution B)} = \frac{\sigma_B}{\bar{X}_B} \times 100 = \frac{10}{45} \times 100 = 22.22\%$$

Since C.V. of (B) > C.V. of (A), the distribution B is more variable than the distribution A. Thus the given statement that variability of wages in factories A & B are same is false.

Karl Pearson's coefficient of skewness for factories A and B is given by

$$\text{Sk (A)} = \frac{(\text{Mean} - \text{Mode})}{\sigma} = \frac{(50 - 45)}{10} = 0.5$$

$$\text{Sk (B)} = \frac{(\text{Mean} - \text{Mode})}{\sigma} = \frac{(45 - 50)}{10} = -0.5$$

Thus the given statement that both the factories have same degree of skewness in respect of wages is false.

**Illustration 71.** In a moderately skewed frequency distribution the mean is ₹30 and the median is ₹20. If the coefficient of variation is 70%, find the Karl Pearson's coefficient of skewness.

**Solution:**

$$\text{Coefficient of variation for the distribution} = \frac{\sigma}{\bar{X}} \times 100 = \frac{\sigma}{30} \times 100 = 70\%$$

$$\text{So standard deviation} = \sigma = 0.7 \times 30 = 21$$

$$\text{Sk} = \frac{3(\text{Mean} - \text{Median})}{\sigma} = \frac{3(30 - 20)}{21} = 1.428$$

**Illustration 72.** Given  $\bar{X} = 20$ , median = 22 and coefficient of skewness = -0.16. Find out Standard deviation.

**Solution:**

$$\text{Karl Pearson's coefficient of skewness} = \text{Sk} = \frac{3(\text{Mean} - \text{Median})}{\sigma} = -0.16$$

$$\text{Or, } \frac{3(20 - 22)}{\sigma} = -0.16$$

$$\text{Or, } \sigma = \frac{3(20 - 22)}{-0.16} = \frac{-6}{-0.16} = 37.5$$

**Illustration 73.** The following facts are gathered before and after an industrial dispute:

Description of items	Before Dispute	After Dispute
No of workers employed	515	509
Mean Wages	₹ 49.5	₹ 52.75
Median Wages	₹ 52.80	₹ 50.00
Variance of wages	Sq. ₹121.00	Sq. ₹ 144.00

Compare the position before and after the dispute in respect of (a) total wages; (b) median wages; (c) modal wages; (d) standard deviation; (e) coefficient of variation; (f) coefficient of skewness.

**Solution:**

(a)

Total Wages in ₹	Mean Wages × No of workers employed	Before dispute	After dispute
		$49.5 \times 515 = 25492.5$	$52.75 \times 509 = 26849.75$

(b) The median wages after the dispute has come down from ₹52.80 to ₹50.00. This implies that before the dispute upper 50% of the workers were getting wages above ₹52.80 whereas after the dispute they get wages only above ₹50.

(c) We know that Mean - Mode = 3 (Mean - Median)

Before Dispute:  $49.5 - \text{Mode} = 3$  ( $49.5 - 52.8$ )

Or,  $\text{Mode} = 49.5 - 3$  ( $49.5 - 52.8$ ) =  $49.5 + 9.9 = ₹ 59.40$

After Dispute:  $52.75 - \text{Mode} = 3$  ( $52.75 - 50.00$ )

Or,  $\text{Mode} = 52.75 - 3$  ( $52.75 - 50.00$ ) =  $52.75 - 8.25 = ₹ 44.50$

It means that there is a concentration of wages around a much smaller value (44.5) after the dispute

(d)

Standard Deviation	Before dispute	After dispute
	$\sqrt{121} = ₹11$	$\sqrt{144} = ₹12$

It means that variability of individual wages increased after the dispute.

(e)

Coefficient of Variation	$\frac{\text{Standard Deviation}}{\text{Mean}} \times 100$	Before dispute	After dispute
		$\frac{11 \times 100}{49.50} = 22.22\%$	$\frac{12 \times 100}{52.75} = 22.75\%$

It means that after dispute more variability of wages are there as  $\text{CV (after)} > \text{CV (before)}$

(f) Coefficient of Skewness:  $Sk = \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}}$

Before dispute:  $Sk = \frac{49.50 - 59.4}{11} = -0.9$

After dispute:  $Sk = \frac{52.75 - 44.5}{12} = 0.6875$

Thus the negatively skewed distributions (before the dispute) has become a positively skewed distribution (after the dispute) which means that the curve of distribution of wages after the dispute has a fairly longer tail towards the right. Hence we conclude that the number of workers getting higher wages has decreased after the dispute.

## EXERCISE PROBLEMS

1. Calculate  $\bar{X}$  for the following data by direct, short cut and step deviation methods.

70	65	55	75	80	85	65	70	95
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**Answer:**

(i) Direct Method:  $\bar{X} = \frac{\Sigma x}{n}$

(ii) Short Cut Method (or) Indirect Method:  $\bar{X} = A + \frac{\Sigma dx}{n}$

(iii) Step Deviation method:  $\bar{X} = A + \frac{\Sigma dx'}{n} \times i$

	$X$	$dx = X - 80$	$dx' = dx/5$
	70	-10	-2
	65	-15	-3
	55	-25	-5
	75	-5	-1
	80	0	0
	85	5	1
	65	-15	-3
	70	-10	-2
	95	15	3
Total	660	-60	-12

Direct Method:  $\bar{X} = \frac{\Sigma x}{n} = \frac{660}{9} = 73.33$

Short Cut Method (or) Indirect Method:  $\bar{X} = A + \frac{\Sigma dx}{n} = 80 - \frac{60}{9} = 73.33$

Step Deviation method:  $\bar{X} = A + \frac{\Sigma dx'}{n} \times i = 80 - \frac{12}{9} \times 5 = 73.33$

2. Calculate  $\bar{X}$  for the following series by Direct, Short-cut and step-deviation Methods.

X	5	10	15	20	25	30	35	40
f	6	17	28	34	18	11	9	7

**Answer:**

(i) Direct Method:  $\bar{X} = \frac{\Sigma fX}{\Sigma f}$

(ii) Short Cut Method (or) Indirect Method:  $\bar{X} = A + \frac{\Sigma fdx}{\Sigma f}$

(iii) Step Deviation method:  $\bar{X} = A + \frac{\Sigma fdx'}{\Sigma f} \times i$

X	f	Xf	dx =X-20	fdx	dx' = dx/5	fdx'
5	6	30	-15	- 90	-3	-18
10	17	170	-10	-170	-2	-34
15	28	420	-5	-140	-1	-28
20	34	680	0	0	0	0
25	18	450	5	90	1	18
30	11	330	10	110	2	22
35	9	315	15	135	3	27
40	7	280	20	140	4	28
Total	130	2675	20	75	4	15

Direct Method:  $\bar{X} = \frac{\sum fX}{\sum f} = \frac{2675}{130} = 20.58$

Short Cut Method (or) Indirect Method:  $\bar{X} = A + \frac{\sum fdx}{\sum f} = 20 + \frac{75}{130} = 20.58$

Step Deviation method:  $\bar{X} = A + \frac{\sum fdx'}{\sum f} \times i = 20 + \frac{15}{130} \times 5 = 20.58$

3. A class has 50 students with average weight of 45 kgs. Out of these there are 30 girls with average weight of 42.5 kgs. Find average wt. of boys.

**Answer:**

$$\bar{X} = \frac{X_1 \times n_1 + X_2 \times n_2 + \dots + X_n \times n_n}{n_1 + n_2 + \dots + n_n}$$

$\bar{X}$  = combined mean of boys & girls = 45 &  $n$  = Total of boys & girls = 50

$\bar{X}_1$  = Mean of girls = 42.5 &  $n_1$  = Total of girls = 30

Total of boys =  $n_2 = n - n_1 = 50 - 30 = 20$

Total weight of boys and girls =  $\bar{X} \times n = 45 \times 50 = 2250$

Total weight of girls =  $\bar{X}_1 \times n_1 = 42.5 \times 30 = 1275$

Total weight of boys = Total weight of boys & girls - Total weight of girls =  $2250 - 1275 = 975$

So Average weight of boys =  $\frac{975}{20} = 48.75$

4. A class of 40 students has an average of 56 marks in Math exam. But later on it was found that terms 48, 54 and 67 were misread as 68, 45 and 87. Find correct mean.

**Answer:** Total marks of 40 students =  $40 \times 56 = 2240$

Correct total =  $2240 - 68 - 45 - 87 + 48 + 54 + 67 = 2209$

So Correct mean =  $\frac{2209}{40} = 55.225$

5.  $\bar{X}$  for 20 items was 36, but two terms were taken as 47 and 56 instead of 67 and 65. Find correct mean.

**Answer:**

Total of 20 students =  $20 \times 36 = 720$

Correct total =  $720 - 47 - 56 + 67 + 65 = 749$

So Correct mean  $\frac{749}{20} = 37.45$

6. Average of 10 terms is 6, Find new average if each term is (i) multiplied by 2 (ii) divided by 5 (iii) 3 is added to each term (iv) 4 is subtracted from each term.

**Hints:**

Total of 10 terms =  $10 \times 6 = 60$

if each term is multiplied by 2, new total will be =  $60 \times 2 = 120$

if each term is divided by 5, new total will be =  $\frac{60}{5} = 12$

if 3 is added to each term, new total will be =  $60 + 10 \times 3 = 90$

if 4 is subtracted from each term, new total will be =  $60 - 10 \times 4 = 20$

So (i)  $\frac{120}{10} = 12$ ; (ii)  $\frac{12}{10} = 1.2$ ; (iii)  $\frac{90}{10} = 9$ ; (iv)  $\frac{20}{10} = 2$ ;

7. Find  $\bar{X}$  for the following series. (Inclusive Series)

Class interval	3-5	6-8	9-11	12-14	15-17	18-20
$f$	3	7	16	34	17	3

**Answer:**

Direct Method:  $\bar{X} = \frac{\sum fx}{\sum f}$ , here  $x$  we take as the mid value of each class

Class	X	f	Xf
3-5	4	3	12
6-8	7	7	49
9-11	10	16	160
12-14	13	34	442
15-17	16	17	272
18-20	19	3	57
	Total	80	992

$$\bar{X} = \frac{\sum fx}{\sum f} = \frac{992}{80} = 12.4$$

8. Find median

Class Interval	Less than 50	50-75	75-100	100-125	125-150	More than 150
f	21	47	67	89	55	21

**Answer:**

$$M_d = l_1 + \frac{\frac{N}{2} - C}{f} \times h$$

Where  $l_1$  = Real lower limit of the median class

$$N = \Sigma f$$

$C$  = Cumulative frequency just preceding the median class

$f$  = simple frequency of the median class

$h$  = width of the median class

Class	f	c.f	Median Class
less than 50	21	21	
50-75	47	68	
75-100	67	135	
100-125	89	224	
125-150	55	279	
More than 150	21	300	
Total	300		

So

$$M_d = l_1 + \frac{\frac{N}{2} - C}{f} \times h = 100 + \frac{\frac{300}{2} - 135}{89} \times 25 = 104.21$$

9. Mean weight of students in a class is 48 kg. If mean weight of girl students is 40 kg. And that of boys is 60 kg. Find (a) % age of boys and girls (b) If there are total of 75 students, find number of each.

**Answer:**

let there are  $x$  no of boy students. So there are  $75 - x$  no of girls

So by question we can write:  $48 = \frac{x \times 60 + (75 - x) \times 40}{75}$

Or,  $75 \times 48 = 60x + 75 \times 40 - 40x$

Or,  $3600 = 20x + 3000$

Or,  $x = \frac{600}{20} = 30$

So no of girls students =  $75 - 30 = 45$

$$\% \text{ age of boys and girls} = \frac{30}{75} \times 100 = 40\% \quad \& \quad \frac{45}{75} \times 100 = 60\%$$

10. Mean of a series with 50 terms is 80; but afterwards it was noted that three terms 63, 47 and 88 were misread as 36, 74 and 63. Find correct Mean.

**Answer:**

$$\text{Total of 50 terms} = 50 \times 80 = 4000$$

$$\text{Correct total} = 4000 - 36 - 74 - 63 + 63 + 47 + 88 = 4025$$

$$\text{So Correct mean} = \frac{4025}{50} = 80.5$$

11. Median of a series is 80, but four terms 38, 66, 93 and 96 were misread as 83, 88, 39 and 69 find correct Median.

12. Calculate M, Q1, Q3 for the following data

21	13	17	11	19	9	16	23	14
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**Answer:**

Arranging the data in ascending order we get

9	11	13	14	16	17	19	21	23
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$$\text{Median} = Md = M = \frac{(n+1)}{2} \text{th term} = \frac{9+1}{2} = 5^{\text{th}} \text{ term} = 16$$

$$Q1 = \frac{(n+1)}{4} \text{th term} = \frac{9+1}{4} = 2.5 \text{th term} = (11 + 13)/2 = 12$$

$$Q3 = \frac{3(n+1)}{4} \text{th term} = \frac{3(9+1)}{4} = 7.5 \text{th term} = (19 + 21)/2 = 20$$

13. Calculate M, Q1, Q3, D6 and P91 for following data

X:	8	12	20	25	30	40
F:	9	16	28	46	20	10

14. Calculate Median Q1, Q3, D4 and P86 for following data

X:	0-4	4-8	8-12	12-16	16-20	20-24	24-28	28-32
F:	4	17	36	90	123	110	66	14

**Answer:**

Class	f	c.f	
0-4	4	4	
4-8	17	21	
8-12	36	57	
12-16	90	147	Q1 class
16-20	123	270	Median Calss,D4 class
20-24	110	380	Q3 class
24-28	66	446	P84 class
28-32	14	460	

So

$$M_d = l_1 + \frac{\frac{N}{2} - C}{f} \times h = 16 + \frac{\frac{460}{2} - 147}{123} \times 4 = 18.699$$

$$Q_1 = l_1 + \frac{\frac{N}{4} - C}{f} \times h = 12 + \frac{\frac{460}{4} - 57}{90} \times 4 = 14.58$$

$$Q_3 = l_1 + \frac{\frac{3N}{4} - C}{f} \times h = 20 + \frac{\frac{3 \times 460}{4} - 270}{110} \times 4 = 22.72$$

$$D_4 = l_1 + \frac{\frac{4N}{10} - C}{f} \times h = 16 + \frac{\frac{4 \times 460}{10} - 147}{123} \times 4 = 17.203$$

$$P_{84} = l_1 + \frac{\frac{84N}{100} - C}{f} \times h = 24 + \frac{\frac{84 \times 460}{100} - 380}{66} \times 4 = 24.388$$

15. Find M, Q1, Q3 for the following figures

X:	4-7	8-11	12-15	16-19	20-23	24-27	28-31
F:	7	11	23	47	36	29	17

16. Calculate D1, M1, Q3 and P95 for the following data (Unequal Intervals)

Class Interval:	0-20	20-30	30-80	80-120	120-180	180-200
Frequency:	7	19	38	78	45	13

17. Find the value of missing frequencies if Q1 = 36 and N = 840

X:	20-25	25-30	30-35	35-40	40-45	45-50	50-55	55-60
F:	40	50	?	150	180	130	?	60

**Hints:**

Let the missing figures are  $f_1$  and  $f_2$

Class	$f$	$cf$	
20-25	40	40	
25-30	50	90	
30-35	$f_1$	$90 + f_1$	120
35-40	150	$240 + f_1$	$Q_1$ class
40-45	180	$420 + f_1$	
45-50	130	$550 + f_1$	
50-55	$f_2$	$550 + f_1 + f_2$	
55-60	60	$610 + f_1 + f_2$	

So by question we can write:

$$610 + f_1 + f_2 = 840, \text{ Or, } f_1 + f_2 = 230$$

$$\text{Rank of } Q_1 = \frac{N}{4} = \frac{840}{4} = 210 \text{ i.e. } 210^{\text{th}} \text{ rank in } Q_1.$$

$$Q_1 = 36 = l_1 + \frac{\frac{N}{4} - C}{f} \times h = 35 + \frac{\frac{840}{4} - 90 - f_1}{150} \times 5 = 35 + \frac{120 - f_1}{30} \text{ So } f_1 = 90$$

$$\text{So from, } f_1 + f_2 = 230, f_2 = 140$$

18. Find out median and quartiles for the following data:

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of Stu	11	18	25	28	30	33	22	15	12	10

19. Calculate P22 and D8

X:	20-29	30-39	40-49	50-59	60-69	70-79
F:	2	19	20	21	15	13

20. In the batch of 15 students, 5 students failed. The marks of 10 students who passed were 9, 6, 7, 8, 8, 9, 6, 5, 4, 7 what were the median marks of all 15 students?

## Exercise:

## Theoretical Questions

## ⊙ Multiple Choice Questions (MCQ)

- Which one of the following is a Positional Average?
  - Geometric Mean;
  - Harmonic Mean;
  - Mode;
  - Progressive Average;
- Because of heavy rain on Sunday average rainfall of a city for the week increased to 0.6 inch from the average rainfall 0.3 inch measured from Monday to Saturday. The rainfall on Sunday was-
  - 2.4 inch;
  - 0.3 inch;
  - 2.1 inch;
  - 1.5 inch
- The mean of the frequency distribution  $((x_1, f_1), (x_2, f_2), (x_3, f_3), \dots, (x_n, f_n))$  is
  - $\frac{\Sigma x}{\Sigma f}$ ;
  - $\frac{\Sigma x}{n}$ ;
  - $\frac{\Sigma fx}{n}$ ;
  - $\frac{\Sigma fx}{\Sigma f}$ ;
- $\sum_{x=1}^{20} x = 54120$ ; While computing this, it was observed that two entries were wrongly entered as 850 and 320 instead of 580 and 230. Correct value of  $\bar{x}$  is
  - 2688;
  - 2746.5;
  - 2720;
  - 2662;

5.  $\Sigma(X - \bar{X})$  is always equal to
- (a) 1;
  - (b) -1;
  - (c) 0;
  - (d)  $\infty$ ;
6. The sum of the squares of deviations of a set of observations is the minimum when deviations are taken from the
- (a) Geometric Mean;
  - (b) Harmonic Mean;
  - (c) Arithmetic Mean;
  - (d) Mode;
7. If each of the values of a variable  $x$  with mean  $\bar{x}$ , is multiplied by  $K$  then the new mean of the variable is
- (a)  $\frac{\bar{x}}{k}$ ;
  - (b)  $k\bar{x}$ ;
  - (c)  $\bar{x}$ ;
  - (d)  $\overline{kx}$ ;
8. Assumed mean is 35,  $\Sigma fd = -425$  &  $\Sigma f = 63$ .  $\bar{x}$  is
- (a) 20;
  - (b) 25.87;
  - (c) 28.25;
  - (d) 19.34;
9. If  $A = 500$ ,  $i = 1000$  &  $\Sigma fd' = 232$  &  $\Sigma f = 120$   $\bar{x}$  is
- (a) 2433.33;
  - (b) 527.84;
  - (c) 501.93;
  - (d) 1017.24;
10. The mean daily salary paid to all employees in a certain company was ₹600. The mean daily salaries paid to the male and female employees were ₹620 and ₹520 respectively. Male to female employees ratio in the company is
- (a) 3:2;
  - (b) 4:5;

- (c) 5:7;
- (d) 4:1;

11. The pass result of 50 students who took up a class test is given below:

Marks	4	5	6	7	8	9
No of Students	8	10	9	6	4	3

If the average marks for all the fifty students was 5.16, the average marks of the students who failed is

- (a) 0.42;
  - (b) 3.06;
  - (c) 4.74;
  - (d) 2.1;
12. For a certain frequency table which has only been partly reproduced below for which the mean is 1.46

No of accidents	0	1	2	3	4	5
Frequency	46	$f_1$	$f_2$	25	10	5

If  $\Sigma f = 200$  Unknown frequencies are

- (a)  $f_1 = 0, f_2 = 114$ ;
  - (b)  $f_1 = 114, f_2 = 0$ ;
  - (c)  $f_1 = 76, f_2 = 38$ ;
  - (d)  $f_1 = 57, f_2 = 57$ ;
13. The sum of the deviations of a certain number of observations measured from 4 is 72 and the sum of the deviations of the observations from 7 is -3. Mean of the observations is
- (a) 6.88;
  - (b) 25;
  - (c) 3.63;
  - (d) Cannot be ascertained with given data;
14. The mean of a certain number of items is 42. If one more item 64 is added to the data, the mean becomes 44. The no of items in the original data is
- (a) 20;
  - (b) 10;
  - (c) 43;
  - (d) 440;

15. The weighted average from the following observation is ₹46.23.

Price per tonne (₹)	45.60	50.70	?
Tonnes Purchased	135	40	25

Simple average of the observation is

- (a) ₹46.23;  
 (b) ₹46.26;  
 (c) ₹66.63;  
 (d) ₹46.24;
16. In a certain factory a unit of work is completed by A in 4 minutes, by B in 5 minutes, by C in 6 minutes, by D in 10 minutes, and by E in 12 minutes. Average number of units of work completed per minute is
- (a)  $25/4$ ;  
 (b)  $5/48$ ;  
 (c)  $4/25$ ;  
 (d)  $25/48$ ;
17. A person walks 8 km at 4km an hour, 6km at 3km an hour and 4km at 2km an hour. Average speed per hour is
- (a) 0.33;  
 (b) 2;  
 (c) 3;  
 (d) 0.5;
18. It is the most suitable average when it is desired to give greater weight to smaller observations and less weight to larger ones. It is
- (a) AM;  
 (b) HM;  
 (c) GM;  
 (d) Median;
19. Which one of the following is not a feature of Arithmetic Mean (AM)?
- (a) AM is affected very much by extreme values;  
 (b) AM is widely used in the study of qualitative phenomenon;

- (c) AM provides a good basis for comparison;
- (d) AM is rigidly defined so different interpretation by different people are not possible;

20. Which one of the following is a feature of Geometric Mean (GM)?

- (a) GM is affected much by the presence of externally small or large observations;
- (b) GM gives the actual value of the series;
- (c) GM is useful when a given phenomenon has a limit for lower value;
- (d) GM is imaginary if any of the observations is zero;

• **State True or False**

1. The arithmetic mean is always the best measure of central tendency
2. The sum of the deviation of the individual observations from mean is zero
3. In a moderately skewed distribution  $A.M. < G.M. < H.M$
4. The addition of a constant value to each of the values of a series increases the average by the same value of the constant.
5. Average alone is enough to throw light on the main characteristics of a statistical series.
6. Median is a mathematical average
7. The value of median and mode can be determined graphically
8. Combined median can be calculated as in case of arithmetic mean
9. Percentile divides the series in ten equal parts
10. Range is the best measure of Dispersion.
11. Mode is the value that has maximum frequency
12. Mode can be located graphically
13. In a positively skewed distribution  $mode > mean$
14. Mode is a mathematical average
15. A distribution with more than two modes is called multimodal
16. Geometric mean is a positional measure of central tendency
17. Geometric mean is more suitable for dealing with problems of rates and speed
18. Geometric mean can be computed in case of open end series
19. Combined G.M. of two or more series can be calculated
20. The geometric mean is the  $n$ th root of the product of  $n$  items in a given distribution.

⊙ **Short Answer Type Questions**

1. What do you mean by “Measures of Central Tendency”?
2. What are different types of averages?
3. If some AMs are given of different series, how can we find combined  $\bar{X}$  for all of those? Also provide the formula.
4. What do you mean by Weighted Arithmetic Mean? How to calculate it? Give it's formula also.
5. Define –
  - (a) Median
  - (b) Quartiles
  - (c) Lower Quartile
  - (d) Upper quartile
  - (e) Decile
  - (f) Percentile
6. How can we calculate (i) Median (ii) Q1 (iii) Q3 (iv) Decile (v) Percentile in Individual or Discrete Series? Also write steps to calculate.
7. How to calculate the above said measures in a continuous series? Write steps to calculate.
8. What amendments are to be made for
  - (i) Inclusive Series
  - (ii) Cumulative series
  - (iii) Open End series
  - (iv) Unequal Interval Series.
9. What are Positional Value or Partition Measures? What are its various types?
10. Define mode
11. What is the Empirical relation between Mean, Mode and Median? Who invented it?
12. When is empirical formula used to calculate Mode? How is it used?
13. Give the precise definition of G.M. along with the formula of its calculation
14. G.M. is a calculated average. Explain.
15. Give three merits of G.M.
16. Give three demerits of G.M.
17. Give three important properties of G.M.
18. Which formula is used to show the rise in prices or the increase in population?
19. Give meaning of H.M.
20. Give three Merits of H.M.

**Answer:**

⊙ **Multiple Choice Questions (MCQ)**

1	2	3	4	5	6	7	8	9	10
(c)	(a)	(d)	(a)	(c)	(c)	(b)	(c)	(a)	(d)
11	12	13	14	15	16	17	18	19	20
(d)	(c)	(a)	(b)	(b)	(c)	(c)	(b)	(b)	(c)

⊙ **State True or False**

1	2	3	4	5	6	7	8	9	10
F	T	F	T	F	F	T	F	F	F
11	12	13	14	15	16	17	18	19	20
T	T	F	F	T	F	F	F	T	T

# CORRELATION AND REGRESSION

# 6

**This Module includes**

- 6.1 Scatter Diagram**
- 6.2 Karl Pearson's Co-efficient of Correlation**
- 6.3 Regression Analysis**

# CORRELATION AND REGRESSION

## Module Learning Objectives:

After studying this Module, the students will be able to understand –

- ✦ What is meant by Correlation and Regression Analysis in statistics
- ✦ Different methods of Computation of
  - Correlation Coefficients
  - Regressions Coefficients
  - Regression Equations
- ✦ Relation between Correlation and Regression
- ✦ Application of correlation and regression in real life situations

# Scatter Diagram

## 6.1

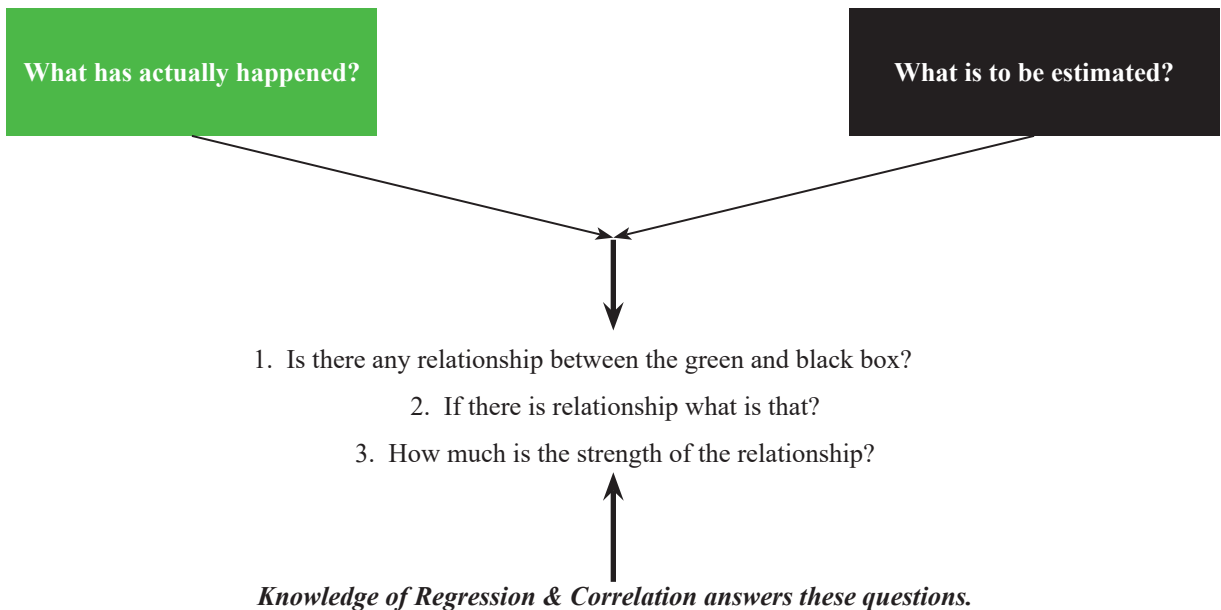
### CORRELATION ANALYSIS

Managers, generally, assess the nature and degree of relationship between variables. A marketing manager likes to know the degree of relationship between advertising expenditure and the volume of sales. A positive relationship is expected between these two variables. The manager is interested in finding out whether there is relationship and if it is what the strength of the relationship is. This expectation is true between two or more variables.

Correlation analysis helps in this regard and Correlation coefficient helps the managers to know the degree of relationship between the variables.

According to A.M. Tuttle, Correlation is defined as an analysis of the covariation between two or more variable.

Now Correlation analysis gives us a statistical tool which helps us to ascertain the degree to which one variable is LINEARLY related to another. Remember for only linear relationship this correlation analysis offer us help to find out degree of association. For other sorts of relationship between variables this will not. All business units in this world make estimate of the future variables on the basis of historical happenings as follows:



**Types of correlation:**

There are three important types of correlation which are discussed as under:

**(I) Positive and Negative Correlation:** Correlation between two variables, positive or negative depends on the direction in which the variables move.

- Positive or direct correlation: If the two variables move in the same direction i.e. with an increase in one variable, the other variable also increases or with a fall in one variable, the other variable also falls, the correlation is said to be positive.
- Negative or Inverse Correlation: If two variables move in opposite direction i.e. with the increase in one variable, the other variable decreases or with the fall in one variable, the other variable rises, the correlation is said to be negative or inverse.

**(II) Simple and Multiple Correlations:**

- Simple Correlation:** When there are only two variables and the relationship is studied between those two variables, it is the case of simple correlation. Relationships between height and weight, price and demand or income and consumption etc. are examples of simple correlation.
- Multiple Correlation:** When there are more than two variables and we study the relationship between one variable and all the other variables taken together then it is a case of multiple correlations. Suppose there are three variables 1, 2, 3 we can study the multiple correlation between A and B & C taken together or between B and A & C together etc. It can be denoted as  $R_{1,23}$  Or  $R_{3,12}$

**(III) Partial and Total Correlation:**

- Partial Correlation:** When there are more than two variables and the relationship between any two of the variables is studied assuming other variables as constant it is a case of partial correlation. This, in fact, is an extension of multiple correlation. Suppose we study the relationship between rainfall and crop, without taking into consideration the effects of other inputs like fertilizers, seeds and pesticides etc., this will be a case of partial correlation. Symbolically if  $x, y, z$  are three variables then partial correlation between  $x$  and  $y$  excluding  $z$  will be given by  $r_{xy.z}$ . Similarly  $r_{xz.y}$  and  $r_{yz.x}$  are possible.
- Total Correlation:** When the correlation among the variables under study is worked out taking all at a time, it is called total correlation.

**Degree and Interpretation of Correlation Coefficient**

According to Karl Pearson, the Coefficient of Correlation lies between  $-1$  &  $+1$  i.e.  $-1 \leq r \leq 1$ . It implies if there is perfect positive relationship between two variables, the value of correlation coefficient would be  $+1$ . On the contrary it would be  $-1$  when there is perfect negative relationship between two variables. Within these limits, the value of correlation coefficient is interpreted as per the table below:

(i) When $r = +1$	Perfect positive correlation
(ii) When $0.75 < r < 1$	High degree of positive correlation
(iii) When $0.5 < r < 0.75$	Moderate degree of positive correlation
(iv) When $0 < r < 0.5$	Low degree of positive correlation
(v) When $r = 0$	No correlation at all
(vi) When $1 < r < -0.75$	High degree of negative correlation

(vii) When $-0.75 < r < -0.5$	Moderate degree of negative correlation
(viii) When $-0.5 < r < 0$	Low degree of negative correlation
(ix) When $r = -1$	Perfect negative correlation

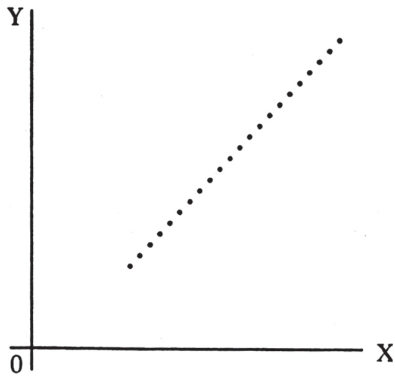
### Scatter Diagram

One of the way of getting an idea about correlation is Scatter Diagram.

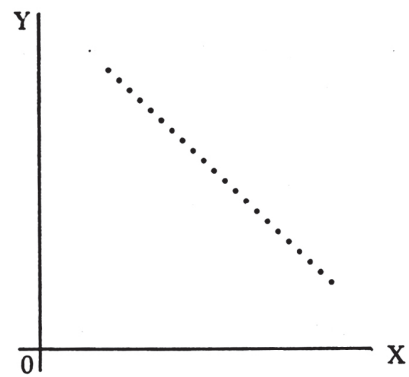
### Scatter Diagram Method

The scatter diagram represents graphically the relation between two variables X and Y. For each pair of X and Y, one dot is put and we get as many points on the graph as the number of observations. Degree of correlation between the variables can be estimated by examining the shape of the plotted dots.

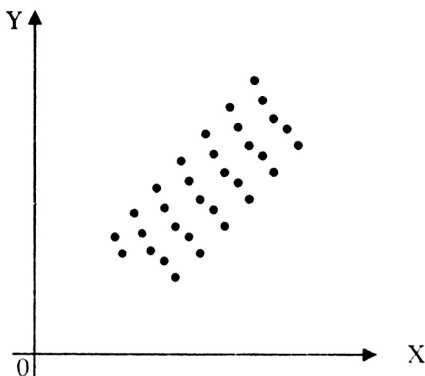
Following are some scattered diagrams showing varied degrees of correlation.



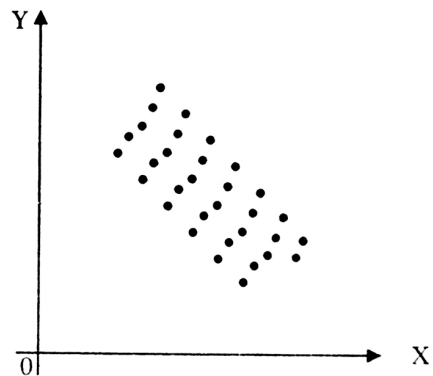
Perfect positive correlation;  $r = 1$



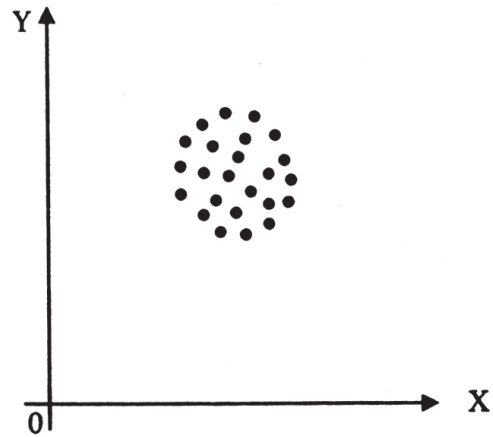
Perfect negative correlation;  $r = -1$



Low degree of positive correlation;  $r > 1$



Low degree of negative correlation;  $r < 1$



No correlation;  $r = 0$

**Advantages**

- (1) It is very easy to draw a scatter diagram
- (2) It is easily understood and interpreted
- (3) Extreme items does not unduly affect the result as such points remain isolated in the diagram

**Disadvantages**

- (1) It does not give precise degree of correlation
- (2) It is not amenable to further mathematical treatment

# Karl Pearson's Co-efficient of Correlation

## METHODS OF STUDY OF CORRELATION

We will study linear correlation through *Karl Pearson's method* or *Covariance method*. Although scatter diagram method and graphic method are there but they are very subjective and rarely used in reality

### Direct method:

$$\text{Coefficient of Correlation} = r = \frac{\Sigma \{(X - \bar{X})(Y - \bar{Y})\}}{\sqrt{\Sigma(X - \bar{X})^2} \sqrt{\Sigma(Y - \bar{Y})^2}} \dots \dots \dots (1)$$

The above formula can also be written as

$$r = \frac{\Sigma \{(X - \bar{X})(Y - \bar{Y})\}}{N\sigma_x\sigma_y} \text{ Or, } r = \frac{\text{Cov}(x, y)}{\sigma_x \cdot \sigma_y} \dots \dots \dots (2)$$

Where  $\sigma_x = \sqrt{\frac{\Sigma(X - \bar{X})^2}{N}}$  and  $\sigma_y = \sqrt{\frac{\Sigma(Y - \bar{Y})^2}{N}}$ .

$\frac{\Sigma \{(X - \bar{X})(Y - \bar{Y})\}}{N}$  is called Cov (x, y) or Co-variance between X & Y.  $\sigma_x$  &  $\sigma_y$  are Standard Deviations of series X and Y.

Another way of writing the above formula is

$$r = \frac{\Sigma XY - \frac{(\Sigma X)(\Sigma Y)}{N}}{\sqrt{\left\{ \Sigma X^2 - \frac{(\Sigma X)^2}{N} \right\} \left\{ \Sigma Y^2 - \frac{(\Sigma Y)^2}{N} \right\}}} \dots \dots \dots (3)$$

### Indirect Method:

$$r = \frac{\Sigma dx \cdot dy - \frac{(\Sigma dx)(\Sigma dy)}{N}}{\sqrt{\left\{ \Sigma dx^2 - \frac{(\Sigma dx)^2}{N} \right\} \left\{ \Sigma dy^2 - \frac{(\Sigma dy)^2}{N} \right\}}} \dots \dots \dots (4)$$

Where  $dx = X - A$ ,  $dy = Y - B$ ,  $A$  &  $B$  are assumed means of  $X$  and  $Y$  series

$$r = \frac{\Sigma dx \cdot dy - N(\bar{X} - A_x)(\bar{Y} - A_y)}{N\sigma_x\sigma_y} \dots \dots \dots (5)$$

Where  $A_x$  &  $A_y$  are assumed means of  $X$  and  $Y$  series

**Illustration 1.** Calculate the Correlation Coefficient between the height of father and son from the given data

Height of father (In inches)	64	65	66	67	68	69	70
Height of son (Inches)	66	67	65	68	70	68	72

**Solution:**

#### Calculation for Correlation Coefficient

Sl. No.	X	Y	$X - \bar{X}$ = X - 67	$Y - \bar{Y}$ = Y - 68	$(X - \bar{X})(Y - \bar{Y})$	$(X - \bar{X})^2$	$(Y - \bar{Y})^2$
1	64	66	-3	-2	6	9	4
2	65	67	-2	-1	2	4	1
3	66	65	-1	-3	3	1	9
4	67	68	0	0	0	0	0
5	68	70	1	2	2	1	4
6	69	68	2	0	0	4	0
7	70	72	3	4	12	9	16
<b>Total</b>	<b>469</b>	<b>476</b>	<b>-</b>	<b>-</b>	<b>25</b>	<b>28</b>	<b>34</b>

Mean of  $X = \bar{X} = \frac{\Sigma X}{n} = \frac{469}{7} = 67$  and Mean of  $Y = \bar{Y} = \frac{\Sigma Y}{n} = \frac{476}{7} = 68$  [ $n =$  No. of pairs of observation = 7]

$$r = \frac{\Sigma \{(X - \bar{X})(Y - \bar{Y})\}}{\sqrt{\Sigma(X - \bar{X})^2} \sqrt{\Sigma(Y - \bar{Y})^2}} = \frac{25}{\sqrt{(28 \times 34)}} = 0.81$$

Using the other formula we can also calculate  $r$  as below:

$$r = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{N\sigma_x\sigma_y} = \frac{25}{7\sqrt{\frac{(X - \bar{X})^2}{N}} \sqrt{\frac{(Y - \bar{Y})^2}{N}}} = \frac{25}{7\sqrt{\frac{28}{7}} \sqrt{\frac{34}{7}}} = 0.81$$

**Alternative Method:**

#### Calculation for Correlation Coefficient

Sl. No.	X	Y	XY	X <sup>2</sup>	Y <sup>2</sup>
1	64	66	4224	4096	4356
2	65	67	4355	4225	4489
3	66	65	4290	4356	4225
4	67	68	4556	4489	4624
5	68	70	4760	4624	4900
6	69	68	4692	4761	4624
7	70	72	5040	4900	5184
<b>Total</b>	<b>469</b>	<b>476</b>	<b>31917</b>	<b>31451</b>	<b>32402</b>

$$r = \frac{\Sigma XY - \frac{(\Sigma X)(\Sigma Y)}{N}}{\sqrt{\left\{ \Sigma X^2 - \frac{(\Sigma X)^2}{N} \right\} \left\{ \Sigma Y^2 - \frac{(\Sigma Y)^2}{N} \right\}}} = \frac{31917 - \frac{(469)(476)}{7}}{\sqrt{\left\{ 31451 - \frac{(469)^2}{7} \right\} \left\{ 32402 - \frac{(476)^2}{7} \right\}}}$$

$$= \frac{25}{\sqrt{28}\sqrt{34}} = 0.81$$

The sign of  $r$  indicates the direction of the relationship between the two variables X and Y. If an inverse relationship exists – that is, if Y decreases as X increases – then  $r$  will fall between 0 and -1. Likewise if there is a direct relationship i.e. if Y increases with increase in X, then  $r$  will fall between 0 and 1.

**Illustration 2.** From the following data compute the co-efficient of correlation between X and Y:

	X series	Y Series
No. of items	15	15
Arithmetic Mean	25	18
Square of deviations from mean	136	138

Summation of product of deviations of X and Y series from their respective Arithmetic Mean is 122.

**Solution:**

$$r = \frac{\Sigma \{(X - \bar{X})(Y - \bar{Y})\}}{\sqrt{\Sigma(X - \bar{X})^2} \sqrt{\Sigma(Y - \bar{Y})^2}} = \frac{122}{\sqrt{136 \times 138}} = 0.89$$

**Illustration 3.** The following table gives the soil temperature and the germination time at various places. Calculate the co-efficient of correlation and interpret the value

Temperature	57	42	40	38	42	45	42	44	40	46	44	43
Germination Time	10	26	30	41	29	27	27	19	18	19	31	29

Take 44 and 26 as assumed means for Temperature and Germination Time respectively.

**Solution:**

$$\text{Coefficient of Correlation} = r = \frac{\Sigma dx \cdot dy - \frac{(\Sigma dx)(\Sigma dy)}{N}}{\sqrt{\left\{ \Sigma dx^2 - \frac{(\Sigma dx)^2}{N} \right\} \left\{ \Sigma dy^2 - \frac{(\Sigma dy)^2}{N} \right\}}}$$

Detail computations are

#### Calculations for Correlation Coefficient

Temperature (X)	Germination (Y)	dx = X-44	dy = Y-26	dx dy	dx <sup>2</sup>	dy <sup>2</sup>
57	10	13	-16	-208	169	256
42	26	-2	0	0	4	0
40	30	-4	4	-16	16	16
38	41	-6	15	-90	36	225
42	29	-2	3	-6	4	9
45	27	1	1	1	1	1
42	27	-2	1	-2	4	1
44	19	0	-7	0	0	49
40	18	-4	-8	32	16	64
46	19	2	-7	-14	4	49
44	31	0	5	0	0	25
43	29	-1	3	-3	1	9
<b>Total</b>	<b>-</b>	<b>-5</b>	<b>-6</b>	<b>-306</b>	<b>255</b>	<b>704</b>

Here  $N =$  No. of pairs of observation = 12

$$r = \frac{\sum dx \cdot dy - \frac{(\sum dx)(\sum dy)}{N}}{\sqrt{\left\{ \sum dx^2 - \frac{(\sum dx)^2}{N} \right\} \left\{ \sum dy^2 - \frac{(\sum dy)^2}{N} \right\}}} = \frac{(-306) - \frac{(-5)(-6)}{12}}{\sqrt{\left\{ 255 - \frac{(-5)^2}{12} \right\} \left\{ 704 - \frac{(-6)^2}{12} \right\}}} = -0.732$$

#### Illustration 4. Given

No. of pairs of observations for X and Y series = 8

X series Arithmetic average = 74.5

X series assumed average = 69

X series standard deviation = 13.07

Y series arithmetic average = 125.5

Y series assumed average = 112

Y series standard deviation = 15.85

Summation of product of corresponding deviations of X and Y series = 2176

Calculate coefficient of correlation.

**Solution:**

$$r = \frac{\Sigma dx.dy - N(\bar{X} - A_x)(\bar{Y} - A_y)}{N\sigma_x\sigma_y}$$

$$\text{So } r = \frac{2176 - 8 \times (74.5 - 69) \times (125.5 - 112)}{8 \times 13.07 \times 15.85} = 0.955$$

**Illustration 5.**

The coefficient of correlation between x and y 0.5, Find coefficient of correlation between x and  $\frac{y}{2}$ .

**Solution:**  $r = 0.5$

Value of r does not change with respect to the change of origin and scale.

**Illustration 6.**

Given correlation coefficient between x and y is 0.5, Find correlation coefficient between u & v, where  $2u - 3x + 4 = 0$  &  $4v + 16y + 11 = 0$

**Solution:** If  $u = \frac{x - a}{b}$ ,  $v = \frac{y - c}{d}$

$$\text{then } r_{xy} = \frac{bd}{|b||d|} r_{uv}$$

Now  $2u - 3x = -4$

Or,  $2u = 3x - 4$

Or,  $u = \frac{3}{2}x - 2$

Or,  $u = \frac{3}{2}(x - \frac{1}{3}) = \frac{2}{3}(b)$

Again  $4v + 16y = -11$

Or  $4v = -16y - 11$  or,  $v = -4y - \frac{11}{4}$

Or  $v = -4(y + \frac{11}{16}) = -4(y - (-\frac{11}{16}))$

Or  $v = \frac{y - \frac{11}{16}}{\frac{1}{4}(d)}$

Now  $r_{xy} = \frac{bd}{|b||d|} r_{uv}$

Or,  $0.5 = \frac{\frac{2}{3} \times (-\frac{1}{4})}{|\frac{3}{5}| |\frac{1}{4}|}$

Or,  $-0.5 = r_{uv}$

**Illustration 7.** If  $R = 0.8$ ,  $\sum D^2 = 33$ , find  $N$ .

**Solution:**

$$R = 1 - \frac{6 \sum D^2}{N^3 - N}$$

Or,  $0.8 = 1 - \frac{6 \times 33}{N^3 - N}$

Or,  $\frac{198}{N^3 - N} = 1 - 0.8 = 0.2$

Or,  $\frac{198}{0.2} = N(N^2 - 1)$

Or,  $990 = (N - 1)N(N + 1)$

Or,  $9 \times 10 \times 11 = (N - 1)N(N + 1)$

Or,  $10 = N$

**Illustration 8.** If correlation Coefficient between  $x$  &  $y$  is  $0.4$ , find 'r' between  $x$  &  $-y$ .

**Solution:**  $-0.4$ .

### Properties of Karl Pearson's coefficient:

The following are the main properties of Karl Pearson's coefficient of correlation.

1. In case correlation is present, then coefficient of correlation would lie between  $\pm 1$  i.e.  $-1 \leq r \leq 1$ . If correlation is absent, then it is denoted by zero.
2. Coefficient of correlation is based on a suitable measure of variation as it takes into account all items of the variable.
3. Coefficient of correlation measures both the direction as well as degree of change in the dependent variable.
4. If there is accidental correlation, in that case the coefficient of correlation might lead to fallacious conclusions. It is known as non-sense or spurious correlation.
5. The coefficient of correlation does not prove causation but it is simply a measure of co-variation. It happens because variations in  $X$  and  $Y$  series may be due to
  - (i) Some common cause,
  - (ii) Some mutual dependence,
  - (iii) Some change and
  - (iv) Some causation of the subject to be relative
6. It is independent of changes of scale and origin of the variables  $X$  and  $Y$ .
7. Coefficient of correlation is the geometric mean of two regression coefficients. Symbolically

$$r = \pm \sqrt{b_{yx} b_{xy}}$$

8. Coefficient of correlation is independent of the unit of measurement
9. Coefficient of correlation works both ways i.e.,  $r_{xy} = r_{yx}$
10. If the value of  $x$  and  $y$  are linearly related with each other i.e., if we have the relation between  $x$  and  $y$  as  $y = ax + b$ , the correlation coefficient between  $x$  and  $y$  will be positive and if the relation between  $x$  and  $y$  be  $y = -ax + b$ , then 'r' will be negative, 'a' being a negative constant.

**Coefficient of Correlation and Probable Error:**

Probable error is an instrument which measures the reliability and dependability of the value of ' $r$ ', the Karl Pearson's coefficient of correlation. The probable error of coefficient of correlation helps in interpreting its value. The coefficient of correlation is generally computed from samples, which are subject to errors of sampling. From the interpretation point of view Probable Error is very useful.

According to Horace Secrist, "The probable error of ' $r$ ' is an amount which if added to and subtracted from the average correlation coefficient, produces amounts within which the chances are even that a coefficient of correlation from a series selected at random will fall".

According to Wheldon, "probable error defines the limit above and below the size of the coefficient determined within which there is an equal chance that the coefficient of correlation similarly calculated from other samples will fall"

Karl Pearson's Probable error is calculated by using the following formula

$$P.E = 0.6745 \frac{1-r^2}{\sqrt{N}}$$

Where P.E. = Probable Error

$r$  = Co-efficient of correlation

$N$  = Number of pairs of observations

This is used in interpreting whether ' $r$ ' is significant or not. If ' $r$ ' is more than P.E. then there is correlation and it is significant if ' $r$ ' is more than six times P.E.

**Utility or Importance of Probable Error:**

Following are the main points of utility of probable error:

- (1) **Determination of Limits:** The Probable error of coefficient of correlation determines the two limits ( $r \pm P.E.$ ) within which, coefficients of correlation of randomly selected samples from the same universe will fall
- (2) **Interpretation of ' $r$ '**

The interpretation of  $r$  based on the probable error is as follows:

- (i) If ' $r$ ' is less than the probable error (i.e.  $r < P.E.$ ), then there is no evidence of correlation in the two variables i.e., correlation is insignificant.
- (ii) If ' $r$ ' is greater than probable error (i.e.  $r > P.E.$ ), then the correlation is significant. If  $r$  is not more than six times the probable error (i.e.  $r \not> 6 P.E.$ ), then it is not significant.
- (iii) If the probable error is small, correlation is definitely existing when  $r > 0.5$
- (iv) If  $r > 0.3$  and probable error is relatively small, ' $r$ ' is not treated as marked
- (v) In other cases nothing can be concluded with certainty

**Illustration 9.** Coefficient of correlation between advertising expenditure and sales for 9 items was observed as +0.69. Find out probable error of correlation coefficient and comment on the significance of  $r$ .

**Solution:**

$$P.E = 0.6745 \frac{1-r^2}{\sqrt{N}} = 0.6745 \times \frac{1-0.69^2}{\sqrt{9}} = 0.118$$

$$r = 0.69 < 6 \times 0.118 = 0.708$$

So, we can say that  $r$  is less than 6 times the value of probable error. Hence the value of  $r$  is not significant.

**Illustration 10.** To study the correlation between the weights and heights of the students of a college, a sample of 100 is taken from the universe. The sample study gives the coefficient of correlation between two variables as 0.9. Within what limits does it hold good for the universe.

**Solution:**

$$\text{Here } r = 0.9, P.E = 0.6745 \frac{1-r^2}{\sqrt{N}} = 0.6745 \times \frac{1-0.9^2}{\sqrt{100}} = 0.013$$

$$\text{Upper limit} = r + P.E = 0.9 + 0.013 = 0.913$$

$$\text{Lower limit} = r - P.E = 0.9 - 0.013 = 0.887$$

**Rank Correlation (Spearman’s Method)**

In 1904, Prof. Charles Edward Spearman had devised a method of computing coefficient of correlation. It is based on the ranking of different items in the variable. This method is useful where actual item values are not given, simply their ranks in the series are known. Thus it is a good measure in cases where abstract quantity of one group is correlated with that of the other group. In Spearman’s coefficient of correlation, we take the differences in ranks, squaring them and finding out the aggregate of the squared differences.

Symbolically

$$r_k = 1 - \frac{6 \sum D^2}{N(N^2 - 1)} \quad \dots \dots \dots (6)$$

Here  $r_k$  = Coefficient of rank correlation

D = Rank differences

N = Number of Pairs

The value of rank correlation falls between  $\pm 1$ . i.e.  $-1 \leq r_k \leq 1$

In rank coefficient of correlation, we shall study three different cases

Case I When ranks against the actual observations are not given

Case II When ranks are given and actual observations may or may not be given

Case III When ranks are equal i.e. against some of the observations of one variable or both the variables have observations with same rank.

**Illustration 11.** Compute Rank correlation coefficient from the data of the following table

X	415	434	420	430	424	428
Y	330	332	328	321	327	325

**Solution:**

This is a problem related to Case I mentioned above.

We will use the following formula

$$r_k = 1 - \frac{6\sum D^2}{N(N^2 - 1)}$$

Detail computations are follows:

#### Calculations for Rank Correlation Coefficient

X	Rank ( $R_1$ )	Y	Rank ( $R_2$ )	D = $R_1 - R_2$	D <sup>2</sup>
415	6	330	3	3	9
434	1	332	1	0	0
420	5	328	4	1	1
430	2	331	2	0	0
424	4	327	5	-1	1
428	3	325	6	-3	9
<b>Total</b>	–	–	–	–	<b>20</b>

Here  $N$  = No. of pairs of observatins = 6

So

$$r_k = 1 - \frac{6\sum D^2}{N(N^2 - 1)} = 1 - \frac{6 \times 20}{6 \times (36 - 1)} = 1 - 0.571 = 0.429$$

**Illustration 12.** The ranks of students in Hindi ( $R_1$ ) and Economics ( $R_2$ ) are given

Hindi ( $R_1$ )	6	1	5	2	4	3
Economics ( $R_2$ )	3	1	4	2	5	6

**Solution:**

This is a problem related to of case II mentioned above

We will use the formula

$$r_k = 1 - \frac{6\sum D^2}{N(N^2 - 1)}$$

Detail computations are as follows:

**Calculations for Rank Correlation Coefficient**

Rank (R <sub>1</sub> )	Rank (R <sub>2</sub> )	D = R <sub>1</sub> - R <sub>2</sub>	D <sup>2</sup>
6	3	3	9
1	1	0	0
5	4	1	1
2	2	0	0
4	5	-1	1
3	6	-3	9
<b>Total</b>	–	–	<b>20</b>

Here N = No. of pairs of observatins = 6

So

$$r_k = 1 - \frac{6 \sum D^2}{N(N^2 - 1)} = 1 - \frac{6 \times 20}{6 \times (36 - 1)} = 1 - 0.571 = 0.429$$

**Case III:** When ranks are equal (Tied or repeated Ranks)

In certain cases, we may find equal ranks for two or more than two values. In that case, each individual item is given an average rank. If two individual items are ranked equal at third place, they are each given the rank (3+4)/ 2 = 3.5, while if three are ranked at third place then (3+4+5)/3 = 4 would be the common rank for third, fourth and fifth places.

When equal ranks are given to some entries, some adjustment in the above formula of Rank Correlation has to be made and the modified formula is:

$$r_k = 1 - \frac{6 \left[ \sum D^2 + \frac{1}{12} \{ (m_1^3 - m_1) + (m_2^3 - m_2) + \dots + (m_n^3 - m_n) \} \right]}{N^3 - N} \dots \dots \dots (7)$$

Where m<sub>1</sub> = 1st Set of Number of items which have common rank, m<sub>2</sub> = 2nd set of Number of items which have common rank .... m<sub>n</sub> = nth set of Number of items which have common rank.

**Illustration 13.** Eight students have obtained the following marks in Accountancy and Economics. Calculate the Rank Co-efficient of correlation.

Accountancy (X)	25	30	38	22	50	70	30	90
Economics (Y)	50	40	60	40	30	20	40	70

**Solution:**

We will use the formula

$$r_k = 1 - \frac{6 \left[ \sum D^2 + \frac{1}{12} \{ (m_1^3 - m_1) + (m_2^3 - m_2) + \dots + (m_n^3 - m_n) \} \right]}{N^3 - N}$$

Where N = No. of pairs of observation = 8

Detail computations are as follows:

## Calculations for Rank Correlation Coefficient

Accountancy (X)	Rank (R <sub>1</sub> )	Economics (Y)	Rank (R <sub>2</sub> )	D = R <sub>1</sub> - R <sub>2</sub>	D <sup>2</sup>
25	7	50	3	4	16
30	5.5	40	5	0.5	0.25
38	4	60	2	2	4
22	8	40	5	3	9
50	3	30	7	-4	16
70	2	20	8	-6	36
30	5.5	40	5	0.5	0.25
90	1	70	1	0	0
<b>Total</b>	–	–	–	–	<b>81.5</b>

[**Note:** While ranking the marks of Accountancy it is seen that there is a tie of 2 students securing the same marks 30 to get placed at the 5th spot. Thus they have given an average rank of  $\frac{5+6}{2} = 5.5$

Similarly in ranking the marks of Economics there is a tie of 3 students securing the marks 40 each to get placed at the 4th spot. Hence they have given an average rank of  $\frac{4+5+6}{3} = 5$ . Also  $m_1 = 2$  and  $m_2 = 3$

So

$$r_k = 1 - \frac{6 \left\{ \sum D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) \right\}}{N^3 - N} = 1 - \frac{6 \left\{ 81.5 + \frac{1}{12}(2^3 - 2) + \frac{1}{12}(3^3 - 3) \right\}}{8^3 - 8} = 0$$

## Coefficient of Correlation (Concurrent Deviation Method)

Under concurrent deviation method only the direction of change in the concerned variables is taken into account. For each term the change is considered with reference to the previous value, which may be either in plus (+) or in minus (-).

Plus (+) is used for showing increase in value & minus (-) is used for showing decrease.

Formula used is

$$r = \pm \sqrt{\left\{ \pm \left( \frac{2C - N}{N} \right) \right\}}$$

Where

$r$  = Correlation coefficient

$C$  = no of concurrent deviations

$N$  = Number of pairs of deviations compared

The meaning of  $\pm$  sign both inside and outside the under root is that the value of  $\left( \frac{2C - N}{N} \right)$  would be positive when

both the signs are positive and it would be negative when both the signs are negative. This is significant in case the value is negative. As because under root of negative values cannot be calculated and therefore if the value of  $\left(\frac{2C - N}{N}\right)$  is negative it is multiplied by the negative (-) sign inside the under root to make it positive. In such case the sign outside will also be minus (-).

**Illustration 14.** Compute the coefficient of correlation for the following data using Concurrent Deviation Method.

Month	Output of Steel (in 000 tons)	Unemployed in steel (in 000)
January	8.5	60
February	9.2	65
March	9.3	61
April	8.5	74
May	7.2	92
June	5.9	157
July	5.1	130
August	6.6	106
September	7.9	58
October	7.6	80
November	8.2	52
December	9.2	45

**Solution:**

Detail computations are

Month	Output	Deviations from preceding month	Unemployed	Deviations from preceding month	Concurrencies	Non Concurrencies
January	8.5		60			
February	9.2	+	65	+	+	
March	9.3	+	61	-		-
April	8.5	-	74	+		-
May	7.2	-	92	+		-
June	5.9	-	157	+		-
July	5.1	-	130	-	+	
August	6.6	+	106	-		-
September	7.9	+	58	-		-
October	7.6	-	80	+		-
November	8.2	+	52	-		-
December	9.2	+	45	-		-
Total					2	9

Concurrencies mean when signs are square i.e. Case 1: deviation of output from preceding month is positive and deviation of unemployed from preceding month is also positive (February), Case 2: deviation of output from preceding month is negative and deviation of unemployed from preceding month is also negative (July). In other cases non-concurrencies are there. So here  $C = 2$  &  $N = 11$ .

So

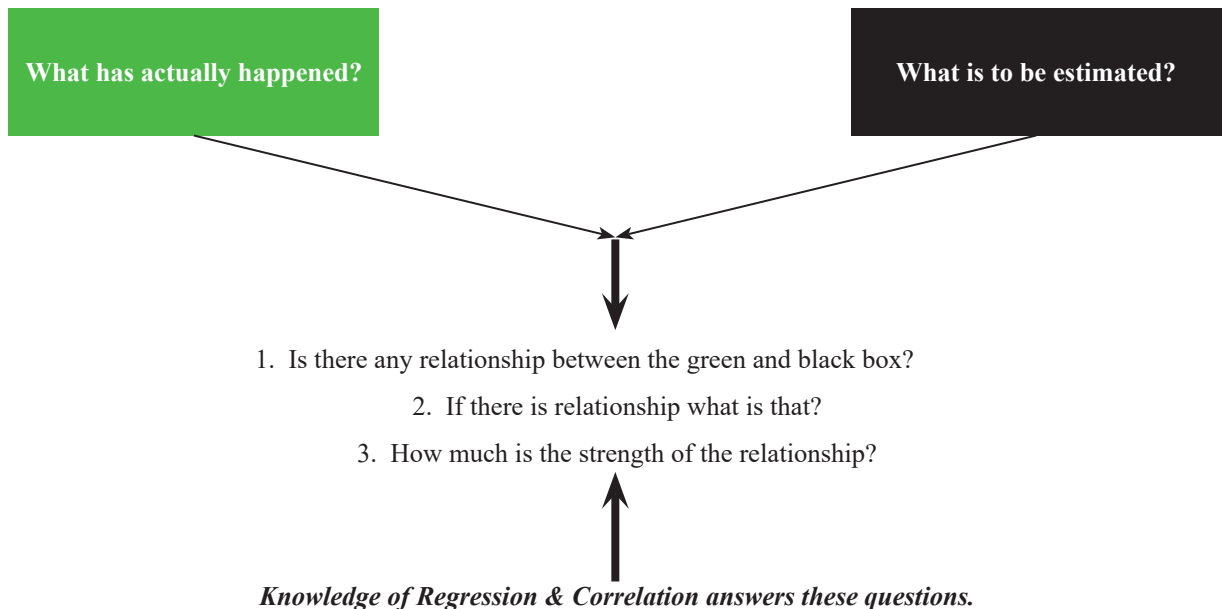
$$r = \pm \sqrt{\left\{ \pm \left( \frac{2C - N}{N} \right) \right\}} = -\sqrt{\left\{ - \left( \frac{2 \times 2 - 11}{11} \right) \right\}} = -0.798$$

# Regression Analysis

## 6.3

### REGRESSION ANALYSIS

Professional Business decisions are based on predictions of future events. To have this predictions, managers got to rely on the relationship between what is already known and what is to be estimated. Knowledge on Regression and Correlation helps managers in this regard. i.e.



### Knowledge of Regression & Correlation answers these questions.

Through regression analysis we, business managers, predict variable represented in the black box, called dependent variable on the basis of another variable represented in the green box, called independent variable.

The statistical technique of estimating or predicting the unknown value of a dependent variable from the known value of an independent variable is called Regression Analysis.

“Regression is the measure of the average relationship between two or more variables in terms of the original units of the data.” – is the definition of Regression according to *Morris M. Blair*.

Sir Francis Galton introduced the concept of 'Regression' for the first time in 1877 when he studied the relationship between the heights of fathers and sons.

### Objectives of Regression analysis:

Regression analysis does the following

1. Explaining the variations in the dependent variable as a result of using a number of independent variables.
2. Describes the nature of relationship in a precise manner by means of regression equation.
3. It is used in prediction and forecasting problems
4. It helps in removing unwanted factors.

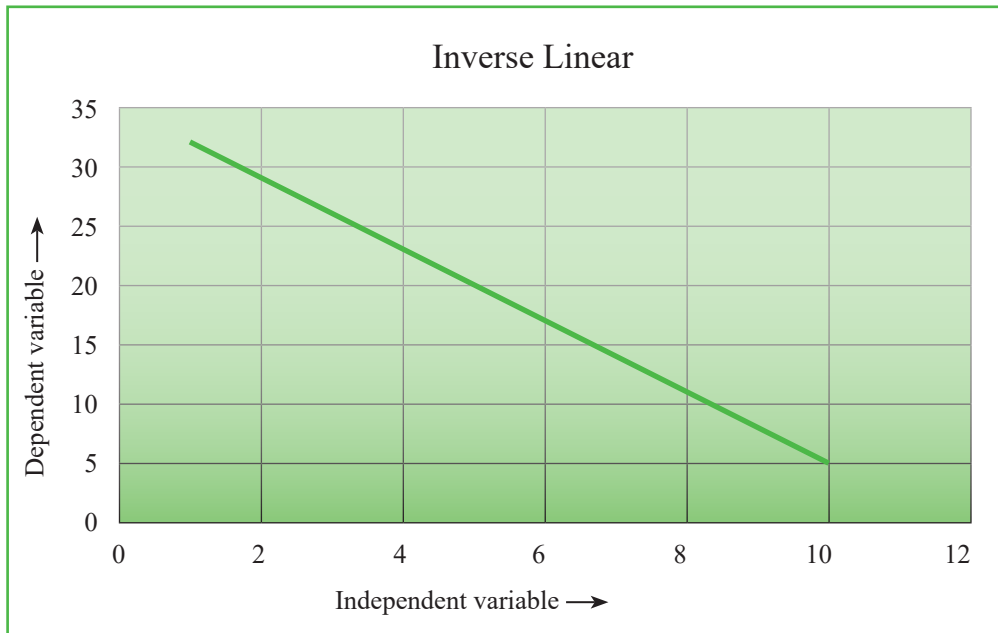
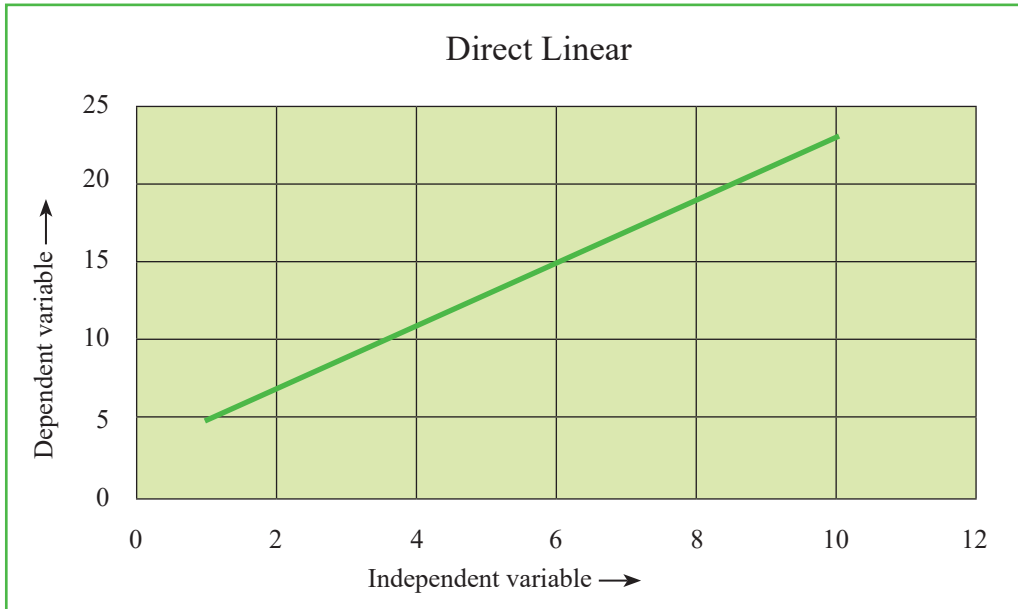
Basically in regression analysis we develop "Regression Equation" which relates independent variables with dependent variables.

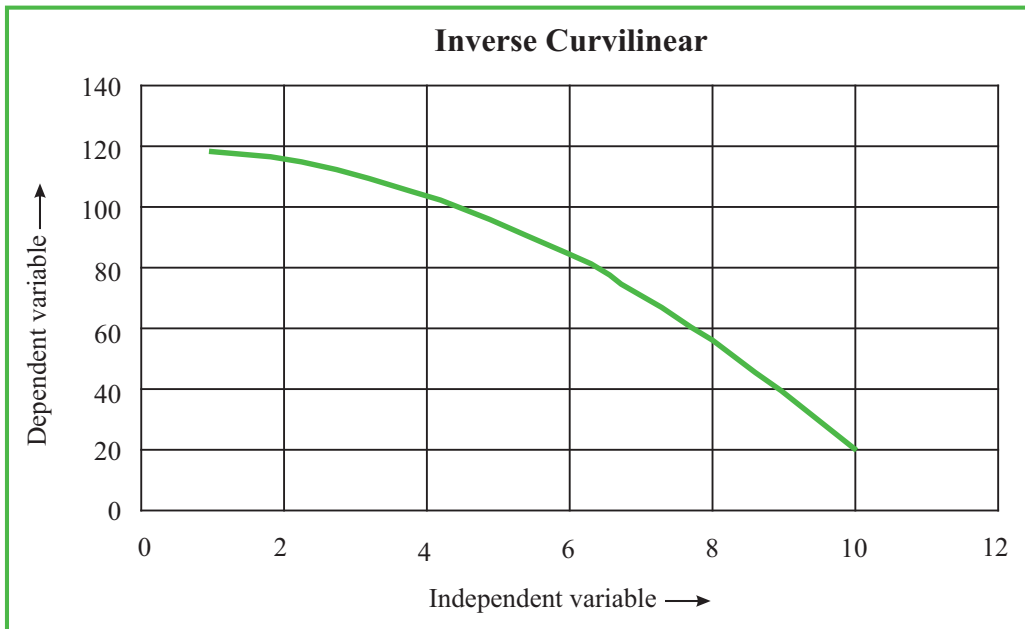
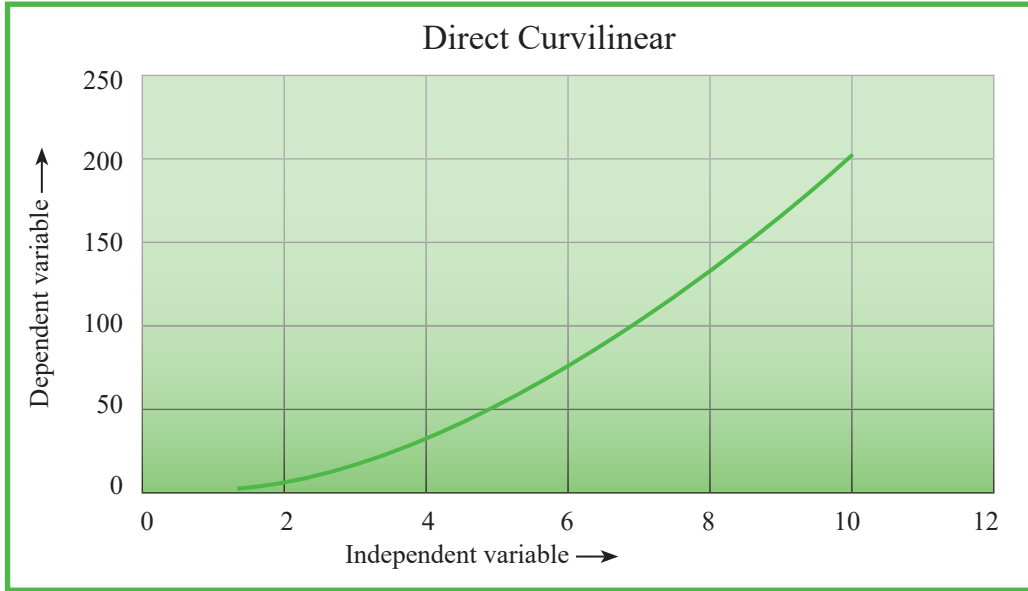
### Classification of Regression Analysis

The regression analysis can be classified on the following basis: -

- (i) Number of variables
  - (ii) Change in Proportion
- (i) **On the basis of Number of Variables:** On the basis of number of variables regression analysis can be classified as –
1. Simple regression
  2. Partial Regression
  3. Multiple Regression
- If Regression equation relates one dependent variable with one independent variable, then Regression is called *Simple Regression*.
- When more than two variables are studied in a functional relationship but the relationship of only two variables is analysed at a time, keeping other variables as constant, such a regression analysis is called *Partial Regression*.
- If Regression equation relates one dependent variable with more than one independent variable, then Regression is called *Multiple Regression*.
- (ii) **On the basis of Change in Proportions:** On the basis of proportions the regression can be classified into the following categories:
1. Linear regression
  2. Non-linear regression

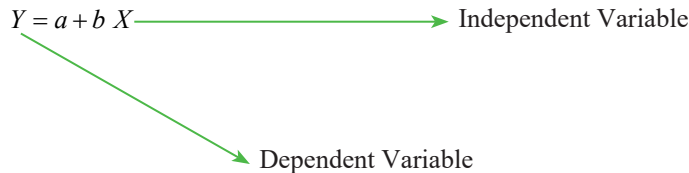
Relationship between variable can be of different types:





Relationship observed by regression is only the relationship of association of variables but not necessarily of cause and effect. Sale of Petrol may have direct relationship with the sale of petrol cars but positive growth of economy actually causes increase in demand of petrol and car.

Linear Regression line is the equation that linearly relates the two variables mathematically. Regression line showing linear relationship between two variables is of the form:



$a$  is called Y axis intercept or Y-intercept or simply intercept,  $b$  is the slope of the line.

A regression line is a graphic technique to show the functional relationship between the two variables X and Y. i.e. dependent and independent variables. It is a line which shows average relationship between two variables X and Y. Thus, this is a line of average. This is also called an estimating line as it gives the average estimated value of dependent variable (Y) for any given value of independent variable (X).

The regression lines can be drawn by two methods as given below: -

1. Free Hand Curve Method
2. The method of Least Squares

1. **Free Hand Curve Method:** This method is also known as the method of Scatter Diagram. This is a very simple method of constructing regression lines. At the same time it is a crude and very rough and rarely used method of drawing regression lines. In this method, the value of paired observations of the variable are plotted on the graph paper. It takes the shape of a diagram scattered over the graphic range of X axis and Y axis. The independent variable is taken on the horizontal axis.

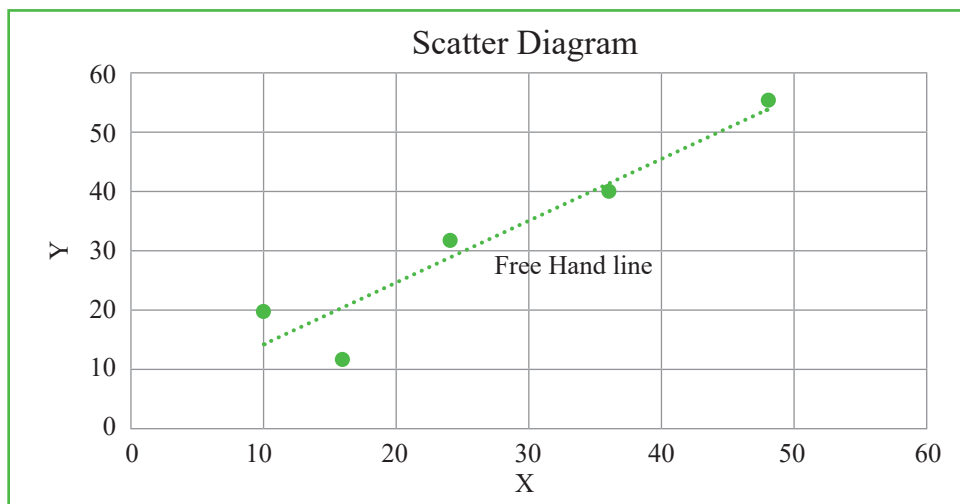
A straight line is drawn through the scattered points on the graph such that it conforms to the following requisites:

- (a) It is as maximum as possible nearer to all the points on the graph
- (b) It is as equidistant from of all the points on either sides of the line
- (c) It passes through the centre of the scattered points.

**Illustration 15.** From the following data relating to X and Y, draw a regression line of Y on X by free hand curve method.

X: 10, 16, 24, 36, 48; Y: 20, 12, 32, 40, 55;

**Solution:**



This method being most subjective is not commonly used in practice.

## 2. The method of Least Squares

The other method of drawing a line of regression is the method of least squares. According to the least squares method the line should be drawn through the plotted points in such a way that the sum of the squares of the deviations of the actual Y values from the computed 'Y' values is the minimum or the least. The line which fits the points in the best manner should have  $\Sigma (Y - Y_c)^2$  as minimum. A line fitted by this method is called the line of best fit. One of the characteristics of the line of best fit is that deviations above the line are equal to the deviations below the line. It implies that the total of the positive and negative deviations is Zero, i.e.  $\Sigma(Y - Y_c) = 0$ . The line of best fit or the straight line passes through the overall mean of the data i.e.  $\bar{X}, \bar{Y}$ .

In the above diagram actual observation of Y for different values of X are plotted and represented by scattered points. The linear regression line which we will form through least square method will follow the equation  $Y = a + bX$  as already mentioned. On development of such a line, if we put different values of X given in the previous illustration we will get different values of Y. These values of Y will be called Computed Y values or in short  $Y_c$ . There could be differences between observed value Y and computed value  $Y_c$ . The difference could be positive or negative, i.e.  $Y - Y_c > 0$  Or  $Y - Y_c < 0$ . Through least square method we actually want to make square of deviations i.e. square of  $(Y - Y_c)$  as minimum or least.

For forming regression line of the form  $Y = a + bX$ , a and b are found out by the method of least squares and the required formulae are:

$$b = \frac{\Sigma XY - n \bar{X} \bar{Y}}{\Sigma X^2 - n \bar{X}^2} \quad \& \quad a = \bar{Y} - b \bar{X}$$

$\bar{X}$  &  $\bar{Y}$  are the mean of respective series.

**Illustration 16.** Using the data in the table below:

X	20	11	15	10	19	21
Y	5	15	14	17	10	11

- (i) Fit the regression line;
- (ii) Predict Y for X = 12;

**Solution:**

### Calculations for Regression Line

Sl. No.	X	Y	XY	X <sup>2</sup>
1	20	5	100	400
2	11	15	165	121
3	15	14	210	225
4	10	17	170	100
5	19	10	190	361
6	21	11	231	441
<b>Total</b>	<b>96</b>	<b>72</b>	<b>1066</b>	<b>1648</b>

Mean of X =  $\Sigma X/n = 96/6 = 16$  and Mean of Y =  $\Sigma Y/n = 72/6 = 12$

$$\bar{X}^2 = (16)^2 = 256$$

$$\begin{aligned} \text{So } b &= \frac{\Sigma XY - n \bar{X} \bar{Y}}{\Sigma X^2 - n \bar{X}^2} = \frac{(1066 - 6 \times 16 \times 12)}{(1648 - 6 \times 256)} \\ &= (-) 86/112 = (-) 0.768 \end{aligned}$$

$$\text{So } a = \bar{Y} - b \bar{X} = 12 - (-) 0.768 \times 16 = 24.288$$

So the regression line is  $Y = 24.288 - 0.768 X$

**Alternative Method of finding ‘a’ and ‘b’**

Let the equation of the Regression line be

$$Y = a + b X$$

Normal equations are

$$\Sigma Y = a \cdot N + b \cdot \Sigma X \quad \dots (1)$$

and  $\Sigma XY = a \Sigma X + b \Sigma X^2 \quad \dots (2)$

Putting the values in (1) and (2) we have

$$72 = a \cdot 6 + b \cdot 96 \text{ or, } a + 16b = 12 \quad \dots (3) \text{ [} N = \text{No. of pairs of observation} = 6 \text{]}$$

$$\text{and } 1066 = a \cdot 96 + b \cdot 1648$$

$$\text{or, } 1066 = (12 - 16b) 96 + 1648b \quad \text{[As from (3), } a = 12 - 16b \text{]}$$

$$\text{or, } 1066 = 1152 - 1536b + 1648b \text{ or, } -8 = 112b \text{ or, } b = -0.768$$

$$\therefore a = 12 - 16b = 12 - 16 \times (-0.768) = 24.288$$

$$\text{If } X = 12 \text{ then } Y = 24.288 - 0.768 \times 12 = 15.072$$

This way of finding regression equation is called forming *Regression Equations through Normal Equations*.

Basically the value of *a* & *b* above are found by simultaneously solving the following two normal equations:

$$\Sigma Y = Na + b \Sigma X \quad \dots \dots \dots (i)$$

$$\Sigma XY = a \Sigma X + b \Sigma X^2 \quad \dots \dots \dots (ii)$$

Equation (i) and (ii) are called normal equations

There are two types of Regression equations:

- (i) Regression equation of Y on X, where Y is the dependent variable and X is the independent one. Equation is of the form  $Y = a + b X$  discussed above.
- (ii) Regression equation of X on Y, where X is the dependent variable and Y is the independent one. Equation is of the form  $X = a + b Y$

For the first case (already discussed) and the second case the related formulae are given below for convenience:

$Y = a + bX$	$X = a + bY$
$b_{yx} = \frac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma X^2 - n\bar{X}^2}$	$b_{xy} = \frac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma Y^2 - n\bar{Y}^2}$
$a = \bar{Y} - b\bar{X}$	$a = \bar{X} - b\bar{Y}$

The two normal equations for regression line X on Y are:

$$\Sigma X = Na + b\Sigma Y \quad \dots \dots \dots \text{(i)}$$

$$\Sigma XY = a\Sigma Y + b\Sigma Y^2 \quad \dots \dots \dots \text{(ii)}$$

Regression equations can be written with the help of *Regression Coefficients* also:

$b_{yx}$ = Regression coefficient	$b_{xy}$ = Regression coefficient
Form of Regression line (Y on X) from Regression Coefficient: $Y - \bar{Y} = b_{yx}(X - \bar{X})$	Form of Regression line (X on Y) from Regression Coefficient: $X - \bar{X} = b_{xy}(Y - \bar{Y})$
$b_{yx} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(X - \bar{X})^2}$	$b_{xy} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(Y - \bar{Y})^2}$

**Illustration 17.** From the following data obtain the two regression equations:

Sales	91	97	108	121	67	124	51	73	111	57
Purchase	71	75	69	97	70	91	39	61	80	47

**Solution:**

**Calculations for Regression Coefficients**

Sales (X)	$(X - \bar{X})$	$(X - \bar{X})^2$	Purchase (Y)	$(Y - \bar{Y})$	$(Y - \bar{Y})^2$	$(X - \bar{X})(Y - \bar{Y})$
91	1	1	71	1	1	1
97	7	49	75	5	25	35
108	18	324	69	-1	1	18
121	31	961	97	27	729	837
67	-23	529	70	0	0	0
124	34	1156	91	21	441	714
51	-39	1521	39	-31	961	1209
73	-17	289	61	-9	81	153
111	21	441	80	10	100	210
57	-33	1089	47	-23	529	759
$\Sigma$	0	6360		0	2868	3900

$$\bar{X} = \frac{\Sigma X}{10} = 900/10 = 90, \bar{Y} = \frac{\Sigma Y}{10} = 700/10 = 70$$

$$\text{So } b_{yx} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(X - \bar{X})^2} = 3900/6360 = 0.613$$

$$\text{So } b_{xy} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(Y - \bar{Y})^2} = 3900/2868 = 1.36$$

So Regression equation  $Y$  on  $X$  is:

$$(i) \quad Y - \bar{Y} = b_{yx}(X - \bar{X})$$

$$\text{Or, } Y - 70 = 0.613(X - 90)$$

$$\text{Or, } Y = 0.613X + 14.83$$

So Regression equation  $X$  on  $Y$  is:

$$(ii) \quad X - \bar{X} = b_{xy}(Y - \bar{Y})$$

$$\text{Or, } X - 90 = 1.36(Y - 70)$$

$$\text{Or, } X = 1.36Y - 5.2$$

**Some important relations:**

1. Regression coefficient of  $x$  on  $y = b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y}$ , where  $r$  = Correlation Coefficient,  $\sigma_x$  = S.d of  $x$  and  $\sigma_y$  = S.d of  $y$
2. Regression coefficient of  $y$  on  $x = b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$ , where the symbols have same meaning as above
3. From (1) and (2) above it can be written that  $r^2 = b_{yx} \cdot b_{xy}$  or,  $r = \pm \sqrt{b_{yx} \cdot b_{xy}}$
4. From (3) above, conclusion can be drawn that Coefficient of determination ( $r^2$ ) is the product of two regression coefficients.
5. Two Regression lines intersect at the point  $(\bar{x}, \bar{y})$
6. To get Regression Coefficient from a given Regression equation the procedure is –
  - (i) If the given Regression Equation is for  $Y$  on  $X$  then convert it to the form  $Y = a + bx$  and ‘ $b$ ’ i.e. the coefficient of  $X$  is the Regression Coefficient  $Y$  on  $X$  or  $b_{yx}$ . In other words  $b_{yx} = b$
  - (ii) If the given Regression Equation is for  $X$  on  $Y$  then convert it to the form  $X = a' + b'Y$  and ‘ $b'$ ’ i.e. the coefficient of  $Y$  is the Regression Coefficient  $X$  on  $Y$  or  $b_{xy}$ . Thus  $b_{xy} = b'$  in this case.

**Illustration 18.** Given the bivariate data:

X	1	5	3	2	1	2	7	3
Y	6	1	0	0	1	2	1	5

Find regression equations by taking deviations of items from the means of  $X$  and  $Y$  respectively

**Solution:**

Regression equations are:

$$Y - \bar{Y} = b_{yx}(X - \bar{X}) \text{ or, } y = b_{yx} \cdot x$$

$$X - \bar{X} = b_{xy}(Y - \bar{Y}) \text{ or, } x = b_{xy} \cdot y$$

Here  $x = X - \bar{X}$  and  $y = Y - \bar{Y}$

$$b_{yx} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(X - \bar{X})^2} = \frac{\Sigma xy}{\Sigma x^2} \dots (1)$$

$$b_{xy} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(Y - \bar{Y})^2} = \frac{\Sigma xy}{\Sigma y^2} \dots (2)$$

Detail computations are as follows:

**Calculations for Regression Coefficients**

Sl. No.	X	Y	$x = X - \bar{X} = X - 3$	$y = Y - \bar{Y} = Y - 2$	$x^2$	$y^2$	$xy$
1	1	6	-2	4	4	16	-8
2	5	1	2	-1	4	1	-2
3	3	0	0	-2	0	4	0
4	2	0	-1	-2	1	4	2
5	1	1	-2	-1	4	1	2
6	2	2	-1	0	1	0	0
7	7	1	4	-1	16	1	-4
8	3	5	0	3	0	9	0
Total	24	16	-	-	30	36	-10

A. M of  $X = \bar{X} = \frac{\Sigma X}{n} = \frac{24}{8} = 3$  and A.M of  $Y = \bar{Y} = \frac{\Sigma Y}{n} = \frac{16}{8} = 2$  [ $n =$  No. of pairs of observation = 8]

So  $b_{yx} = \frac{-10}{30} = \frac{-1}{3}$  &  $b_{xy} = \frac{-10}{36} = \frac{-5}{18}$  [Putting the values in (1) and (2)]

So  $Y - \bar{Y} = b_{yx}(X - \bar{X})$  Or,  $Y - 2 = \frac{-1}{3} \times (X - 3)$  Or  $X + 3Y - 9 = 0$

and

$X - \bar{X} = b_{xy}(Y - \bar{Y})$  Or,  $X - 3 = \frac{5}{18}(Y - 2)$  Or  $18X + 5Y - 64 = 0$

**Alternatively:**

$$r = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\sqrt{\Sigma(X - \bar{X})^2} \sqrt{\Sigma(Y - \bar{Y})^2}} = \frac{-10}{\sqrt{30 \times 36}} = -0.304$$

$$\sigma_x = \sqrt{\frac{30}{8}} = 1.94 \quad \text{and} \quad \sigma_y = \sqrt{\frac{36}{8}} = 2.12$$

Regression equation of  $X$  on  $Y$ :

$$X - \bar{X} = b_{yx}(Y - \bar{Y}) = X - 3 = -0.304 \times \frac{1.94}{2.12}(Y - 2)$$

$$\text{Or, } X - 3 = -0.304 \times \frac{1.94}{2.12}(Y - 2) = -0.277(Y - 2)$$

$$\text{Or, } X - 3 = -0.277Y + 0.556$$

$$\text{Or, } X + 0.277Y - 3.556 = 0$$

Regression equation of  $Y$  on  $X$ :

$$Y - \bar{Y} = b_{xy}(X - \bar{X}) = Y - 2 = -0.304 \times \frac{2.12}{1.94}(X - 3)$$

$$\text{Or, } Y - 2 = -0.304 \times \frac{2.12}{1.94}(X - 3) = -0.333(X - 3)$$

$$\text{Or, } Y - 2 = -0.333X + 0.999$$

$$\text{Or, } 0.333X + Y - 2.999 = 0$$

**The Standard error of Estimate:**

To measure the reliability of the regression equation (i.e. degree of scattering of observed values of dependent variable around the computed value of dependent variable from the regression line i.e.  $(Y - Y_c)$  a measuring parameter named Standard error of Estimate is to be used. The formula for finding

$$S_e = \sqrt{\frac{\sum Y^2 - a \sum Y - b \sum XY}{n - 2}}$$

Larger the Standard error of estimate, the greater the scattering of points around the regression line. If  $S_e = 0$  then all the given values of  $Y$  will lie exactly on the regression line for the given values of  $X$ .

For the observed points assuming normally distributed around the regression line we can expect

- (i) 68% of the points within  $\pm 1S_e$  ;
- (ii) 95.5% of the points within  $\pm 2S_e$  ;
- (iii) 99.7% of the points within  $\pm 3S_e$  ;

**Illustration 19.** A study by the University, Department of Transportation on the effect of bus ticket prices upon the number of passengers produced the following results:

Ticket price (cents)	15	20	25	30	40	50
No. of Passengers per 100 Miles	440	430	450	370	340	370

- i) Develop the regression equation;
- ii) Determine Standard Error of Estimate;
- iii) Interpret the result;

**Solution:**

Let the Regression Equation be  $Y = a + bX$

**Calculations for Regression Equation**

Sl. No.	Ticket Price (cents) (X)	No. of Passenger/100 miles (Y)	XY	X <sup>2</sup>	Y <sup>2</sup>
1	15	440	6600	225	193600
2	20	430	8600	400	184900
3	25	450	11250	625	202500
4	30	370	11100	900	136900
5	40	340	13600	1600	115600
6	50	370	18500	2500	136900
Total	180	2400	69650	6250	970400

A. M of  $X = \bar{X} = \frac{\Sigma X}{n} = \frac{180}{6} = 30$  and A.M of  $Y = \bar{Y} = \frac{\Sigma Y}{n} = \frac{2400}{6} = 400$  [ $n =$  No. of pairs of observation = 6]

$$\text{So } b = \frac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma X^2 - n\bar{X}^2} = \frac{(69650 - 6 \times 30 \times 400)}{(6250 - 6 \times 900)} \\ = -2.765$$

$$\text{So } a = \bar{Y} - b\bar{X} = 400 - (-2.765) \times 30 = 482.95$$

Required Regression equation is  $Y = 482.95 - 2.765X$

$$\text{So } S_e = \sqrt{\frac{\Sigma Y^2 - a \Sigma Y - b \Sigma XY}{n-2}} \\ = \sqrt{\frac{970400 - 482.95 \times 2400 - (-2.765) \cdot 69650}{6-2}} = 31.234$$

Interpretation:

- (i) 68% of the points are within  $\pm S_e$  i.e.  $\pm 31.234$ ;
- (ii) 95.5% of the points are within  $\pm 2S_e$  i.e.  $\pm 62.468$ ;
- (iii) 99.7% of the points are within  $\pm 3S_e$  i.e.  $\pm 93.702$

$Y = a + bX$	$X = a + bY$
$S_e = \sqrt{\frac{\Sigma Y^2 - a \Sigma Y - b \Sigma XY}{n-2}}$	$S_e = \sqrt{\frac{\Sigma X^2 - a \Sigma X - b \Sigma XY}{n-2}}$

**Illustration 20.** For certain X and Y series which are correlated, the two lines of regression are:

$5X - 6Y + 90 = 0$  and  $15X - 8Y - 130 = 0$ . Find the means of the two series and the correlation coefficient between them.

**Solution:** The two regression line intersects each other at the point  $(\bar{X}, \bar{Y})$ .

From first equation we have  $X = \frac{1}{5}(6Y - 90) = 1.2Y - 18$

Putting this value in 2<sup>nd</sup> equation we have  $15(1.2Y - 18) - 8Y - 130 = 0$

$$\text{Or } 18Y - 270 - 8Y - 130 = 0$$

$$\text{Or } 10Y = 400$$

$$\text{Or } Y = 40$$

$$\text{So } X = 1.2 \times 40 - 18 = 30$$

Therefore Means of the two series are  $\bar{X} = 30, \bar{Y} = 40$

Let us suppose 1<sup>st</sup> equation is the equation of line of regression of Y on X and the 2<sup>nd</sup> equation is that of X on Y

So from 1<sup>st</sup> equation we can write  $6Y = 5X + 90$

$$\text{Or, } Y = \frac{5}{6}X + \frac{90}{6}$$

So  $b_{YX}$  = Coefficient of X in the Regression Equation Y on X =  $\frac{5}{6}$

Similarly from 2<sup>nd</sup> equation we can write  $15X = 8Y + 130$

$$\text{Or, } X = \frac{8}{15}Y + \frac{130}{15}$$

So  $b_{XY}$  = Coefficient of Y in the Regression Equation X on Y =  $\frac{8}{15}$

Therefore  $r^2 = b_{YX} \cdot b_{XY} = \frac{5}{6} \cdot \frac{8}{15} = \frac{40}{90}$  or,  $r = \pm \frac{2}{3}$

Since both the regression coefficients are positive, r must be positive and we take  $r = 2/3$

[**Note:** If we had assumed that the 1<sup>st</sup> equation is the regression equation of X on Y and 2<sup>nd</sup> equation is the regression equation of Y on X we get  $b_{XY} = \frac{6}{5}$  &  $b_{YX} = \frac{15}{8}$

Therefore  $r^2 = b_{YX} \cdot b_{XY} = \frac{6}{5} \cdot \frac{15}{8} = \frac{90}{40}$  Or,  $r = \frac{3}{2} > 1$  which not a feasible solution]

**Illustration 21.** The regression equation of profit (X) on sales (Y) of a certain firm is  $3Y - 5X + 108 = 0$ . The average sales of the firm was ₹44,000 and the variance of profits is 9/16<sup>th</sup> of the variance of sales. Find the average profit and the coefficient of correlation between sales and profit.

**Solution:**

The regression equation of profit (X) on sales (Y) of the firm is  $3Y - 5X + 108 = 0$ .

Average sales  $\bar{Y} = 44,000$

The average profit can be obtained by putting  $Y = 44000$  in the regression equation of X on Y (as  $\bar{X}, \bar{Y}$  satisfy both the regression equations)

We get  $3 \times 44000 - 5X + 108 = 0$ , Or  $132000 + 108 = 5X$

Or,  $X = \frac{132108}{5} = 26421.6$ . So required value of Average Profit =  $\bar{X} = ₹ 26421.60$

By question  $\sigma_x^2 = \frac{9}{16}\sigma_y^2$  or  $\frac{\sigma_x}{\sigma_y} = \frac{3}{4}$

From the equation  $3Y - 5X + 108 = 0$  we can write  $X = \frac{3}{5}Y + \frac{108}{5}$

$b_{xy}$  = Coefficient of  $Y$  in the Regression equation  $X$  on  $Y$

$$= \frac{3}{5} = r \frac{\sigma_x}{\sigma_y}$$

So  $r = \frac{3}{5} \cdot \frac{4}{3} = \frac{4}{5} = 0.8$

**Illustration 22.** While calculating the coefficient of correlation between two variable  $X$  and  $Y$  the following results were obtained:

The number of observations  $N = 25$ ,  $\Sigma X = 125$ ,  $\Sigma Y = 100$ ,  $\Sigma X^2 = 650$ ,  $\Sigma Y^2 = 460$ ,  $\Sigma XY = 508$ . It was however later discovered at the time of checking that two pairs of observations  $(X, Y)$  were copied as  $(6, 14)$  and  $(8, 6)$  while the correct values were  $(8, 12)$  and  $(6, 8)$  respectively. Find the correct value of the coefficient of correlation.

**Solution:**

$$\text{Corrected } \Sigma X = 125 - 6 - 8 + 8 + 6 = 125$$

$$\text{Corrected } \Sigma Y = 100 - 14 - 6 + 12 + 8 = 100$$

$$\text{Corrected } \Sigma X^2 = 650 - 6^2 - 8^2 + 8^2 + 6^2 = 650$$

$$\text{Corrected } \Sigma Y^2 = 460 - 14^2 - 6^2 + 8^2 + 12^2 = 436$$

$$\text{Corrected } \Sigma XY = 508 - 6 \times 14 - 8 \times 6 + 8 \times 12 + 6 \times 8 = 520$$

Correct value of Coefficient of Correlation

$$\begin{aligned} r_{Cov} &= \frac{(\Sigma XY)_{Cov} - [(\Sigma X)_{Cov} \cdot (\Sigma Y)_{Cov}] / N}{\sqrt{\left\{ (\Sigma X^2)_{Cov} - \frac{(\Sigma X)^2_{Cov}}{N} \right\} \left\{ (\Sigma Y^2)_{Cov} - \frac{(\Sigma Y)^2_{Cov}}{N} \right\}}} \\ &= \frac{520 - [125 \times 100] / 25}{\sqrt{\left\{ 650 - \frac{(125)^2}{25} \right\} \left\{ 436 - \frac{(100)^2}{25} \right\}}} \\ &= \frac{520 - 500}{\sqrt{\{650 - 625\} \{436 - 400\}}} = \frac{20}{\sqrt{25 \times 36}} = \frac{20}{5 \times 6} = \frac{20}{30} = \frac{2}{3} \end{aligned}$$

**Illustration 23.** You are given the following data.

	X	Y
Arithmetic Mean	20	25
Standard Deviation	5	4

Coefficient of correlation between X and Y is 0.6. Form the regression equations of X on Y and Y on X

**Solution:**

We know that

Regression coefficient of Y on X =  $b_{XY} = \frac{r \sigma_y}{\sigma_x}$  and

Regression coefficient of X on Y =  $b_{YX} = \frac{r \sigma_x}{\sigma_y}$

So  $b_{YX} = \frac{0.6 \times 4}{5} = 0.48$  &  $b_{XY} = \frac{0.6 \times 5}{4} = 0.75$

Regression equation of Y on X:  $Y - \bar{Y} = b_{YX} (X - \bar{X})$

$$Y - 25 = 0.48 (X - 20)$$

$$\text{Or, } Y - 25 = 0.48X - 9.6$$

$$\text{Or, } Y = 0.48X + 15.4$$

Regression equation of X on Y:  $X - \bar{X} = b_{XY} (Y - \bar{Y})$

$$X - 20 = 0.75 (Y - 25)$$

$$\text{Or, } X - 20 = 0.75Y - 18.75$$

$$\text{Or, } X = 0.75Y + 1.25$$

**Illustration 24.** Given the bivariate data:

X: 2, 6, 4, 3, 2, 2, 8, 4; Y: 7, 2, 1, 1, 2, 3, 2, 6;

(a) Fit the regression line of Y on X and hence predict Y, if X = 20

(b) Fit the regression line of X on Y and hence predict X, if Y = 5.

**Solution:**

Regression equations are:

$$Y - \bar{Y} = b_{YX} (X - \bar{X})$$

$$X - \bar{X} = b_{XY} (Y - \bar{Y})$$

$$b_{yx} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(X - \bar{X})^2}$$

$$b_{xy} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(Y - \bar{Y})^2}$$

### Calculations for Regression Coefficients

Sl. No.	X	Y	$x = X - \bar{X} = X - 3.875$	$y = Y - \bar{Y} = Y - 3$	xy	$x^2$	$y^2$
1	2	7	-1.875	4	-7.5	3.515625	16
2	6	2	2.125	-1	-2.125	4.515625	1
3	4	1	0.125	-2	-0.25	0.015625	4
4	3	1	-0.875	-2	1.75	0.765625	4
5	2	2	-1.875	-1	1.875	3.515625	1
6	2	3	-1.875	0	0	3.515625	0
7	8	2	4.125	-1	-4.125	17.015625	1
8	4	6	0.125	3	0.375	0.015625	9
Total	31	24			-10	32.875	36

Mean of  $X = \bar{X} = \frac{\Sigma X}{n} = \frac{31}{8} = 3.875$ , Mean of  $Y = \bar{Y} = \frac{\Sigma Y}{n} = \frac{24}{8} = 3$  [ $n = \text{No. of pairs of observation} = 8$ ]

$$b_{yx} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(X - \bar{X})^2} = -\frac{10}{32.875} = -0.304$$

$$b_{xy} = \frac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(Y - \bar{Y})^2} = -\frac{10}{36} = -\frac{5}{18} = -0.278$$

So the Regression line  $Y$  on  $X$  is

$$Y - \bar{Y} = b_{yx}(X - \bar{X}) \text{ Or, } Y - 3 = -0.304(X - 3.875) \text{ Or, } 0.304X + Y - 4.178 = 0,$$

When  $X = 20$  then  $0.304 \times 20 + Y - 4.178 = 0$  Or,  $Y = -1.902$

So the Regression line  $X$  on  $Y$  is

$$X - \bar{X} = b_{xy}(Y - \bar{Y}) \text{ Or, } X - 3.875 = -0.278(Y - 3) \text{ Or, } X + 0.278Y - 4.709 = 0,$$

When  $Y = 5$  then  $X + 0.278 \times 5 - 4.709 = 0$  Or,  $X = 3.319$

**Illustration 25.**

Find 'r' from the regression line  $2x + 3y = 5$ .

**Solution:** Regression line of y or x.

$$2x + 3y = 5$$

$$\text{Or, } 3y = -2x + 5$$

$$\text{Or, } y = -\frac{2}{3}x + \frac{5}{3}, \quad \text{Hence } b_{yx} = -\frac{2}{3}$$

$$\text{Or, } r \frac{\sigma_y}{\sigma_x} = -\frac{2}{3} \quad (\sigma = \text{S.D})$$

$$\text{Or, } r = -1. \quad (\sigma_y, \sigma_x \text{ are +ve})$$

**Illustration 26.**

If  $b_{yx} = 0.2$ ,  $b_{xy} = 0.3$  &  $\sigma_x = 2$ , find  $\sigma_y$ .

$$\text{Solution: } \frac{b_{yx}}{b_{xy}} = \frac{\sigma_y^2}{\sigma_x^2} \quad \text{or, } \frac{0.2}{0.3} = \frac{\sigma_y^2}{4}$$

$$\text{Or } \sigma_y^2 = \frac{0.8}{0.3} = 2.66 \quad \text{or } \sigma_y = 1.63$$

**Illustration 27.** Find  $b_{yx}$  from the regression line.

$$7x - 3y - 18 = 0.$$

$$\text{Solution: } b_{yx} = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{7}{-3} = \frac{7}{3}$$

**Illustration 28.** Find  $b_{xy}$  from the regression line

$$4x - y - 11 = 0.$$

$$\text{Solution: } b_{xy} = -\frac{\text{coefficient of } y}{\text{coefficient of } x} = -\frac{-1}{4} = \frac{1}{4}$$

**Properties of Regression Coefficient:**

The main properties of regression coefficients are as under:

- Both the regression coefficients  $b_{xy}$  and  $b_{yx}$  cannot be greater than unity, i.e., either both are less than unity or one of them must be less than unity. In other words, the square root of the product of two regression coefficients must be less than or equal to 1 or  $-1$  or  $\sqrt{b_{xy} \times b_{yx}} \leq 1$
- Both the regression coefficients will have the same sign i.e.,
  - If  $b_{xy}$  is positive, then  $b_{yx}$  will also be positive.
  - If  $b_{xy}$  is negative, then  $b_{yx}$  will also be negative.
  - Both  $b_{xy}$  and  $b_{yx}$  must have same signs. If both are positive, r will be positive and vice-versa.

3. Correlation Co-efficient is the geometric mean between regression coefficients i.e.,  $r = \pm \sqrt{b_{xy} \times b_{yx}}$
4. A.M of Regression Coefficients is greater than or equal to the Coefficient of correlation i.e.  $\frac{b_{xy} + b_{yx}}{2} \geq r$
5. Regression Coefficients are independent of change of origin but not of scale.

**Illustration 29.** Past data on household income and expenditure reveals that

- (a) The average absolute increase in income in relation to increase in expenditure is ₹1.5 crore and
- (b) The average absolute increase in expenditure in relation to increase in income in ₹50 crore

Find the coefficient of correlation between household income and expenditure.

**Solution:**

Let the Increase in Income =  $x$

And the increase in expenditure =  $y$ .

The value of regression coefficient  $b_{xy} = 1.5$

Similarly the regression coefficient  $b_{yx} = 0.5$

$$r = \pm \sqrt{b_{xy} \times b_{yx}} = +\sqrt{1.5 \times 0.5} = \sqrt{0.75} = 0.866 \text{ [As both } b_{xy} \text{ and } b_{yx} \text{ are positive, } r \text{ is also positive]}$$

### Correlation and Regression:

Correlation and Regression analysis are two important statistical tools to study the functional relationship between the variables. Coefficient of correlation is a measure of degree of covariance between  $x$  and  $y$  whereas the aim of regression analysis is to study the nature of relationship between the variables, so that we may know the value of one variable on the basis of another. However, we can study the relation and difference between the correlation and regression as under.

### Relation between Correlation and Regression:

The regression lines are used to predict the value of dependent variable given the different values of independent variable. Two regression lines show the average relationship between two variables.

If there is perfect correlation (positive or negative, i.e.,  $r \pm 1$ ) both the lines will coincide. In this case there will be one regression line.

In case  $r = 0$ , i.e., both the variables are independent, both the lines will intersect each other at right angles i.e., parallel to the axes.

Higher the value of  $r$ , closer will be the regression lines to each other.

On the other hand if the value of  $r$  is low, the gap between regression lines will be wider.

It is important to note that the point, where both the regression lines intersect each other, gives us the mean values of  $X$  and  $Y$ .

### Difference between Correlation and Regression

The difference between correlation and regression can be analysed as under –

1. **Nature of Relationship:** Correlation analysis tests the closeness of the variables, whereas regression analysis measures the extent of change in dependent (y) variable due to change in the independent (x) variable i.e., both the nature and extent of functional relationship between the variables is studied.
2. **Relationship:** In regression analysis, the causal relationship in variables moving in the same or opposite direction is studied while in correlation analysis, the study is made by taking into consideration the cause and effect relationship between the variables. i.e. only closeness of the variables is studied.
3. **Mutual dependence of variables:** Correlation studies the mutual dependence of variables but in regression analysis the functional relationship showing dependence of one variable upon other is analysed.
4. **Spurious Correlation:** In correlation chance of spurious correlation between two variables, having no practical importance, may be observed, but there is no chance of existence of such type of relation in regression analysis.
5. **Mathematical treatment:** Correlation is having no scope of further mathematical treatment where as regression can be used for further treatment.
6. **Origin & Scale:** Regression analysis is independent of change of origin but not of scale, where as correlation is both independent of change in origin and scale.
7. **Relative and Absolute Measures:** Regression analysis is an absolute measure showing the change in the value of y or x for unit change in the value of x or y. whereas correlation coefficient is a relative measure of linear relationship between x and y and is independent of the measurement. It is a number which lies between +1 & -1.
8. **Applicability:** Correlation has very limited scope of application. It is limited to the linear relationship between two variables, but the scope of applicability of regression analysis is very wide. It can be covered under linear as well as non-linear relationship between the variables.
9. **Differentiation in variables:** In correlation both variables are considered at par for study purposes, whereas in regression analysis variables are differentiated as dependent and independent variables.
10. **Symmetrical or Asymmetrical formation:** Correlation is symmetrical in formation, i.e.  $r_{xy} = r_{yx}$ . The measurement of the co-variability of two variables is symmetrical in formation. It means 'r' is a both way relationship of x on y or y on x, but in regression, the approach is one way of analysis treating one variable as dependent and the other as independent, thus making the analysis Asymmetrical, that is  $b_{xy} \neq b_{yx}$ .

**EXERCISE - I**

1. From the following table calculate the coefficient of correlation by Karl Pearson's method and also fill the gap.

X	8	4	12	6	10
Y	11	13	?	10	9

Arithmetic means of X and Y series are 8 and 10 respectively.

2. Given Number of pairs of observations of X and Y = 9

X Series Arithmetic average = 70.5,

X Series Standard deviation = 9.07,

X series Assumed average = 65.0,

Y Series Assumed average = 108.0

Y Series Arithmetic Average = 121.5

Y Series standard deviation = 11.85

Summation of product of corresponding deviation of X and Y series = 1451

Calculate coefficient of correlation.

3. Find the rank correlation for the following distribution

Marks in Economics	48	60	72	62	56	40	39	52	30
Marks in Accountancy	62	78	65	70	38	54	60	32	31

4. Calculate correlation coefficient from the following results:

$$N = 10, \Sigma x = 100, \Sigma y = 150, \Sigma (X-10)^2 = 180, \Sigma (Y-15)^2 = 215, \Sigma (X-10)(Y-15) = 60$$

5. If the covariance between X and Y variables is 10 and the variance of X and Y are respectively 16 and 9, find the coefficient of correlation.
6. From the data given below, find the number of items.

$$r = 0.5, \Sigma xy = 120, \Sigma x^2 = 90, \sigma_y = 8$$

(Where x and y are deviations from arithmetic average).

7. From the following data compute the coefficient of correlation between X and Y:

	X- Series	Y- Series
Arithmetic Mean	15	28
Sum of Squares of deviations from mean	144	225

Summation of product of deviations of X and Y series from their respective means = 20

8. In a question on correlation the value of r is 0.64 and its P.E. = 0.1312. What was the value of N?
9. From the marks obtained by 8 students in Accountancy and Statistics, compute rank coefficient of correlation.

Marks in Accountancy	60	15	20	28	12	40	80	20
Marks in Statistics	10	40	30	50	30	20	60	30

10. The coefficient of rank correlation between marks in Quantitative Mathematics and Economics obtained by a certain group of students is  $7/11$ . The sum of the squares of differences in ranks is 60. What is the number of students in the group?
11. The coefficient of rank correlation of the marks obtained by 10 students in statistics and accountancy was found to be 0.5. It was later discovered that the difference in ranks in the two subjects obtained by one of the students was wrongly taken as 3 instead of 7. Find the correct coefficient of rank correlation.
12. Find correlation coefficient for data given below:

Age (years)	42	36	48	43	55	52	38
Blood Pressure	132	120	140	133	142	148	122

13. Calculate Karl Pearson's coefficient of correlation between expenditure on advertisement and sales from the data given below:

Advertisement expense in ₹ '000	39	78	65	62	90	82	75	25	98	36
Sales in lakh ₹	47	84	53	58	86	62	68	60	91	51

14. Find the coefficient of correlation and probable error from the data:

X	67	69	71	75	85	93	87	73
Y	95	80	87	80	79	75	80	85

15. From the following data compute the coefficient of correlation between X and Y.

	X- Series	Y- Series
Arithmetic Mean	25	18
Sum of Squares of deviations from mean	136	138

Summation of product of deviations of X and Y series from their respective means = 122. Number of pairs of value is 15.

16. Ten competitors in a beauty contest are ranked by 3 judges in the following order:

Judge A	1	6	5	10	3	2	4	9	7	8
Judge B	3	5	8	4	7	10	2	1	6	9
Judge C	6	4	9	8	1	2	3	10	5	7

Use Rank correlation coefficient to determine which pair of judges has the nearest approach to common taste in beauty.

17. Calculate the coefficient of correlation by any suitable method:

A	115	168	170	127	118	129	135	140
B	2	6	8	11	1	1	4	3

Also calculate probable error

18. From the data given below, compute the correlation coefficient by the method of concurrent deviations.

Year	1996	1997	1998	1999	2000	2001	2002
Supply	150	154	160	172	160	165	180
Price	200	180	170	160	190	180	172

19. Consider the data summarized below:

X	42	44	58	55	89	98	66
Y	56	49	53	58	65	76	58

The coefficient of correlation between the variables X and Y is 0.64, their covariance is 16. The variance of X is 9. Find the standard deviation of Y.

20. Calculate the coefficient of correlation by concurrent deviation method from the following:

X	60	55	50	55	30	70	40	35	80	80	75
Y	65	40	35	75	63	80	35	20	80	60	60

21. Calculate Pearson's coefficient of correlation from the following data using 44 and 26 as the origin of X and Y respectively.

X	43	44	46	40	44	42	45	42	38	40	42	57
Y	29	31	19	18	19	27	27	29	41	30	26	10

22. Given the bivariate data

X	2	6	4	3	2	3	8	4
Y	7	2	1	1	2	3	2	6

Obtain regression equations taking deviations from 5 in case of X and 4 in case of Y.

23. Past 10 years data on Rainfall and Yield of wheat in a certain village offered the following results.

Av. Wheat Yield	25 Qtl
Av. Rainfall	20 Cms
Variance of wheat output	3 Qtl
Variance of Rainfall	5 Cms
Correlation Coefficient	0.65

Find the most likely yield output per acre when rainfall is 35cms.

24. In a partially destroyed laboratory record relating to correlation data, the following results are Legible  $\sigma_x^2 = 9$ , Regression equations  $8X - 10Y + 66 = 0$  &  $40X - 18Y = 214$ .

What were (a) the mean values of X and Y, (b)  $\sigma_y$  (c) the co-efficient of correlation between X and Y?

25. Given

	X- Series	Y- Series
Arithmetic Mean	18	100
Standard Deviation	14	20

Coefficient of correlation between X and Y is + 0.8. Find out

- (a) The most probable value of Y if X is 70 and most probable value of X if Y is 90.  
 (b) If the regression coefficients are 0.8 and 0.6, what would be the value of the coefficient of correlation?
26. Given that the means of X and Y are 65 and 67, their standard deviations are 2.5 and 3.5 respectively and the coefficient of correlation between them is 0.8.
- (i) Write down the equations of the regression lines  
 (ii) Obtain the best estimate of X when Y = 70  
 (iii) Using the estimated value of X as the given value of X, estimate the corresponding value of Y.

27. The correlation coefficient between the variables X and Y is  $r = 0.60$ . If  $\sigma_x = 1.50$ ,  $\sigma_y = 2.00$ ,  $\bar{X} = 10$ ,  $\bar{Y} = 20$ , find the equations of the regression lines (i) Y on X (ii) X on Y.
28. Find out  $\sigma_y$  and  $r$  from the following data:  $3x = y$ ,  $4y = 3x$  and  $\sigma_x = 2$ .
29. Given that the regression equations of Y on X and X on Y are respectively  $Y = X$  and  $4X - Y = 3$ . Find the correlation coefficient between X and Y.
30. From the following data calculate (i) coefficient of correlation (ii) Standard deviation of Y.

$$X = 0.854 Y; Y = 0.89X; \sigma_x = 3$$

31. If the two lines of regression are

$$4X - 5Y + 30 = 0 \text{ and } 20X - 9Y - 107 = 0$$

Which of these is the line of regression of X on Y. Find  $r$  and  $\sigma_y$  when  $\sigma_x = 3$ .

32. From the following regression equations, calculate  $\bar{X}$ ,  $\bar{Y}$  and  $r$

$$20x - 9Y = 107$$

$$4X - 5Y = -33$$

33. The following table gives the age of car of a certain make and annual maintenance costs. Obtain the regression equation for cost related to age, Estimate maintenance cost of a car whose age is 10 Years.

Age of cars (X) in years	2	4	6	8
Maintenance Cost in hundreds of ₹ (Y)	10	20	25	30

34. Find two regression equations from the following data:

X: 10, 25, 34, 42, 37, 35, 36, 45;

Y: 56, 64, 63, 58, 73, 75, 82, 77;

35. Obtain the line of regression of Y on X for the following data:

Age (Years) X: 66, 38, 56, 42, 72, 36, 63, 47, 55, 45;

Blood Pressure Y: 145, 124, 147, 125, 160, 118, 149, 128, 150, 124;

Estimate the blood pressure of a man whose age is 50 years.

36. In trying to evaluate the effectiveness in its advertising campaign, a firm compiled the following information.

Year	1998	1999	2000	2001	2002	2003	2004	2005
Advertisement expenditure (₹'000)	12	15	15	23	24	38	42	48
Sales (₹ lakhs)	5	5.6	5.8	7	7.2	8.8	9.2	9.5

Calculate the regression equation of sales on advertising expenditure. Estimate the probable sales when advertisement expenditure is ₹60 thousand.

37. Two random variables have the least square regression lines with equations  $3x + 2y - 26 = 0$  and  $6x + y - 31 = 0$ . Find the mean value and the correlation coefficient between X and Y.
38. The following data show the years of services (x) and average salary per month (y) with respect to 7 persons working in a college. Fit a straight line regression of Y on X.

Sr. No	1	2	3	4	5	6	7
X (years)	1	3	7	6	2	4	5
Y (₹'000)	5	7	7	8	6	4	3

## Exercise:

## Theoretical Questions

⊙ Multiple Choice Questions (MCQ)

- If  $b_{XY}$  and  $b_{YX}$  are regression coefficients of series X on series Y and regression coefficients of series Y on series X respectively then which one of the following is correct?
  - $b_{XY} \times b_{YX} = r$ , where  $r$  is the correlation coefficient
  - $b_{XY} \times b_{YX} = r^2$ , where  $r$  is the correlation coefficient
  - $b_{XY} \times b_{YX} = -r$ , where  $r$  is the correlation coefficient
  - $b_{XY} \times b_{YX} = 1/r$ , where  $r$  is the correlation coefficient
- If  $b_{XY}$  and  $b_{YX}$  are regression coefficients of series X on series Y and regression coefficients of series Y on series X respectively then which one of the following is correct?
  - $b_{XY}$  and  $b_{YX}$  will be either both positive or both negative
  - $b_{XY}$  will be positive and  $b_{YX}$  will be negative
  - $b_{XY}$  will be negative and  $b_{YX}$  will be positive
  - Nothing can be said like this, it depends on X & Y values

- Say true or false:

“If  $r$  is the correlation coefficient between two series X & Y then covariance between X & Y is given as “ $r \times \text{variance of series X} \times \text{variance of series Y}$ ”

- Match the items in column A with the items in column B

A	B
a) $Y - \bar{Y}$	i) is used to obtain the best estimates of X for given values of Y
b) $X - \bar{X} = b_{XY}(Y - \bar{Y})$	ii) is equal to $r \times \sigma_X$
c) $b_{YX} = \frac{1}{b_{XY}}$	iii) is equal to $b_{YX}(X - \bar{X})$
d) $\sigma_Y \times b_{XY}$	iv) is a condition for making $r^2 = 1$

- Which one of the following is correct?
  - Regression equation predicts maximum probable values of one variable for specified values of other variable
  - Regression equation predicts most likely values of one variable for specified values of other variable
  - Regression equation predicts maxi-min values of one variable for specified values of other variable
  - Regression equation predicts minimum probable values of one variable for specified values of other variable
- If  $r = 0.52$ ,  $\sigma_X = 4.6$  &  $\sigma_Y = 36.8$  then  $b_{XY}$  is equal to
  - 0.24
  - 4.16
  - +1
  - 0.065

7. If  $b_{XY}$  &  $b_{YX}$  are regression coefficients between X on Y and Y on X respectively and  $r$  is the correlation coefficient between X and Y then
- $\frac{b_{XY} + b_{YX}}{2} \leq r$
  - $\frac{b_{XY} + b_{YX}}{2} \leq r^2$
  - $\frac{b_{XY} + b_{YX}}{2} \geq r$
  - $\frac{b_{XY} + b_{YX}}{2} \geq r^2$
8. If  $b_{XY}$  &  $b_{YX}$  are regression coefficients between X on Y and Y on X respectively then
- $\sqrt{b_{XY} \times b_{YX}} \leq 1$
  - $\sqrt{b_{XY} \times b_{YX}} \geq 1$
  - $\sqrt{b_{XY} \times b_{YX}} = -1$
  - $\sqrt{b_{XY} \times b_{YX}} \leq 1$
9. In a regression equation
- Regression coefficient represents the increment in the value of the independent variable for a unit change in the value of the dependent variable
  - Regression coefficient represents the increment in the value of the dependent variable for a unit change in the value of the independent variable
  - Regression coefficient represents the mean value of the independent variable for a unit change in the value of the dependent variable
  - Regression coefficient represents the mean value of the dependent variable for a unit change in the value of the independent variable
10. If  $r^2 = 0.3$  &  $b_{XY} = -1.5$  then  $b_{YX}$  is equal to
- +1
  - 0.2
  - 1
  - 0.45
11. In a bivariate regression analysis  $\Sigma XY = 1355.25$ ,  $(\Sigma X)(\Sigma Y) = 6396$ ,  $\Sigma X^2 = 591.50$  &  $\Sigma X = 52$ . If there are 5 items then  $b_{YX}$
- 1
  - 0.97
  - 0.667
  - 1.5
12.  $X = 1.36Y - 5.2$  &  $Y = 0.61X + 1.51$  are two regression equations. Correlation coefficient between  $X$  &  $Y$  is
- 0.67
  - 0.911
  - 0.911
  - 0.67

13. In a bivariate regression analysis comprising of series X & Y, if  $\Sigma(X - \bar{X})^2 = \Sigma(Y - \bar{Y})^2$  then

- (a)  $b_{XY} = b_{YX}$
- (b)  $b_{XY} > b_{YX}$
- (c)  $b_{XY} < b_{YX}$
- (d) Correlation coefficient = 1

14. Match the items in column A with the items in column B

A	B
a) Coefficient of correlation	i) is a functional relationship which shows dependence of one variable on the other
b) Arithmetic mean of regression coefficients	ii) is the geometric mean between regression coefficients
c) Regression Analysis	iii) is the mean values of the variables X and Y in case of two variables case
d) Intersection point of lines of regression	iv) is greater than equal to correlation coefficient

15. In a bivariate regression analysis the difference between actual value of dependent variable and the predicted value of the dependent variable is called

- (a) Outlier
- (b) Slope
- (c) Residual
- (d) Scattered point

16. Consider the following results:

$$N = 12, \Sigma dx = 0, \Sigma dy = 4, \Sigma dx^2 = 1344, \Sigma dy^2 = 215, \Sigma dxdy = -4360$$

Appropriate regression coefficient of y on x is

- (a) -0.821
- (b) 1
- (c) 5.67
- (d) -3.244

17. In a bivariate regression analysis for dependent variable if  $d = \text{Actual value} - \text{Predicted value}$  then at different values of independent variable

- (a) Best fit curve occurs when  $d_1^2 + d_2^2 + \dots + d_n^2$  is minimum
- (b) Best fit curve occurs when  $d_1^2 + d_2^2 + \dots + d_n^2$  is maximum
- (c) Best fit curve occurs when  $d_1^2 + d_2^2 + \dots + d_n^2$  is zero
- (d) Best fit curve occurs when  $d_1^2 + d_2^2 + \dots + d_n^2$  is one

18. Consider the following results:

$$N = 6, \Sigma y = 42, \Sigma y^2 = 318, b_{yx} = -\frac{11}{34}, \Sigma x^2 - \frac{1}{n}(\Sigma x)^2 = 34$$

Then  $b_{xy}$  is

- (a) 11/34
  - (b) 11/24
  - (c) - 34/11
  - (d) - 11/24
19. In a bivariate analysis if two regression equations are  $mx - y + 10 = 0$  &  $-2x + 5y = 14$ . If coefficient of correlation between  $x$  &  $y$  is  $\frac{1}{\sqrt{10}}$ , then value of  $m$  is
- (a) 10
  - (b) 5/2
  - (c) 4
  - (d) 1
20. In a bivariate analysis if two regression equations are  $8x - 10y + 66 = 0$  &  $40x - 18y - 214 = 0$ . Then  $\bar{x}$ ,  $\bar{y}$ , the mean of the series  $x$  &  $y$  care respectively
- (a) 13, 17
  - (b) 17, 17
  - (c) 5/4, 20/9.
  - (d) 8, 18

**Answer:**

⊙ **Multiple Choice Questions (MCQ)**

1.	2.	3.	4.	5.
(b)	(a)	False	(a) (iii), (b) (i), (c) (iv), (d) (ii)	(b)
6.	7.	8.	9.	10.
(d)	(c)	(a)	(b)	(b)
11.	12.	13.	14.	15.
(d)	(c)	(a)	(a) (ii), (b) (iv), (c) (i), (d) (iii)	(c)
16.	17.	18.	19.	20.
(d)	(a)	(d)	(c)	(a)

# PROBABILITY

# 7

**This Modules includes**

- 7.1 Concepts and Terminologies**
- 7.2 Primary applications of Probability Theorems**

# PROBABILITY

## Module Learning Objectives:

After studying this Module, the students will be able to understand –

- ✦ Principal concepts of Probability
- ✦ Features of Probability
- ✦ Application of Permutation and Combination in solving problems of Probability
- ✦ Sample Space, Theories of Probability
- ✦ Conditional probability, Rule of total probability
- ✦ Bayes' Theorem
- ✦ Application of probability in real world

# Concepts and Terminologies

## 7.1

### GENERAL CONCEPT

The concept of probability is difficult to define in precise terms. In ordinary language, the word probable means likely or chance. The probability theory is an important branch of mathematics. Generally the word, probability, is used to denote the likelihood of the occurrence of an event, based on past experiences. By looking at the clear sky, one will say that there will not be any rain today. On the other hand, by looking at the cloudy sky or overcast sky, one will say that there will be rain today. In the earlier sentence, we aim that there will not be rain and in the latter we expect rain. On the other hand a mathematician says that the probability of rain is 0 in the first case and that the probability of rain is 1 in the second case. In between 0 and 1, there are fractions denoting the chance of the event occurring.

If a coin is tossed, the coin falls down. The coin has two sides ; head and tail. On tossing a coin, the coin may fall down either with the head up or tail up. It is assumed that a coin, on reaching the ground, will not stand on its edge. So the probability of the coin coming down with either Head or Tail up is 1. The probability of the head coming up is 50% and the tail coming up is 50%, In other words we can say the probability of the head or the tail coming up is  $\frac{1}{2}$ , i.e ‘head’ and ‘tail’ share equal chances. The probability that it will come down head or tail is unity.

### SOME USEFUL TERMS

Before discussing the theory of probability, let us have an understanding of the following terms :

#### Random Experiment or Trial

If an experiment or trial can be repeated under the same conditions, any number of times and it is possible to count the total number of outcomes, but individual result i.e. individual outcome is not predictable. Suppose we toss a coin. It is not possible to predict exactly the outcomes. The outcome may be either head up or tail up. Thus an action or an operation which can produce any result or outcome is called a random experiment or a trial.

#### Outcome

This means the possible result of a Random Experiment. If we consider the Random Experiment of “Tossing of a Coin” then there are two possible results – appearance of Head or H and appearance of Tail or T. Hence the outcomes in this case are H & T.

#### Event

Any possible outcome of a random experiment is called an event. Performing an experiment is called trial and outcomes are termed as events.

An event whose occurrence is inevitable when a certain random experiment is performed, is called a sure event or certain event. At the same time, an event which can never occur when a certain random experiment is performed is called an impossible event. The events may be simple or composite. An event is called simple if it corresponds to a

single possible outcome. For example, in rolling a die, the chance of getting 2 is a simple event. Further in tossing a die, chance of getting event numbers (1, 3, 5) are compound event.

### Sample space

The set or aggregate of all possible outcomes is known as sample space. For example, when we roll a die, the possible outcomes are 1, 2, 3, 4, 5, and 6 ; one and only one face come upwards. Thus, all the outcomes—1, 2, 3, 4, 5 and 6 are sample space. And each possible outcome or element in a sample space called sample point.

### Mutually exclusive events or cases

Two events are said to be mutually exclusive if the occurrence of one of them excludes the possibility of the occurrence of the other in a single observation. The occurrence of one event prevents the occurrence of the other event. As such, mutually exclusive events are those events, the occurrence of which prevents the possibility of the other to occur. All simple events are mutually exclusive. Thus, if a coin is tossed, either the head can be up or tail can be up; but both cannot be up at the same time.

Similarly, in one throw of a die, an even and odd number cannot come up at the same time. Thus two or more events are considered mutually exclusive if the events cannot occur together.

### Equally likely events

The outcomes are said to be equally likely when one does not occur more often than the others.

That is, two or more events are said to be equally likely if the chance of their happening is equal. Thus, in a throw of a die the coming up of 1, 2, 3, 4, 5 and 6 is equally likely. For example, head and tail are equally likely events in tossing an unbiased coin.

### Exhaustive events

The total number of possible outcomes of a random experiment is called exhaustive events. The group of events is exhaustive, as there is no other possible outcome. Thus tossing a coin, the possible outcome are head or tail; exhaustive events are two. Similarly throwing a die, the outcomes are 1, 2, 3, 4, 5 and 6. In case of two coins, the possible number of outcomes are 4 i.e. (22), i.e., HH, HT TH and TT. In case of 3 coins, the possible outcomes are  $2^3=8$  and so on. Thus, in a throw of n" coin, the exhaustive number of case is  $2^n$ .

### Independent Events

A set of events is said to be independent, if the occurrence of any one of them does not, in any way, affect the Occurrence of any other in the set. For instance, when we toss a coin twice, the result of the second toss will in no way be affected by the result of the first toss.

### Dependent Events

Two events are said to be dependent, if the occurrence or non-occurrence of one event in any trial affects the probability of the other subsequent trials. If the occurrence of one event affects the happening of the other events, then they are said to be dependent events. For example, the probability of drawing a king from a pack of 52 cards is  $4/52$ , ; the card is not put back; then the probability of drawing a king again is  $3/51$ . Thus the outcome of the first event affects the outcome of the second event and they are dependent. But if the card is put back, then the probability of drawing a king is  $4/52$  and is an independent event.

### Simple and Compound Events

When a single event take place, the probability of its happening or not happening is known as simple event.

When two or more events take place simultaneously, their occurrence is known as compound event (compound probability); for instance, throwing a die.

### Complementary Events

The complement of an event, means non-occurrence of A and is denoted by  $\bar{A}$ . It contains those points of the sample space which do not belong to A. For instance let there be two events A and B.  $\bar{A}$  is called the complementary event of A and vice versa, if A and B are mutually exclusive and exhaustive.

### Favourable Cases

The number of outcomes which result in the happening of a desired event are called favourable cases to the event. For example, in drawing a card from a pack of cards, the cases favourable to “getting a diamond” are 13 and to “getting an ace of spade” is only one. Take another example, in a single throw of a dice the number of favourable cases of getting an odd number are three - 1, 3 and 5.

### MEASUREMENT OF PROBABILITY

The origin and development of the theory of probability dates back to the seventeenth century. Ordinarily speaking the probability of an event denotes the likelihood of its happening. A value of the probability is a number ranges between 0 and 1. Different schools of thought have defined the term probability differently. The various schools of thought which have defined probability are discussed briefly.

#### Classical Approach (Priori Probability)

The classical approach is the oldest method of measuring probabilities and has its origin in gambling games. According to this approach, the probability of an event is the ratio of favourable outcomes to the total number of mutually exclusive, exhaustive and equally likely outcomes. If we toss a coin we are certain that the head or tail will come up. The probability of the coin coming down is 1, of the head coming up is  $\frac{1}{2}$  and of the tail coming up is  $\frac{1}{2}$ . It is customary to describe the probability of one event as ‘ $p$ ’ (success) and of the other event as ‘ $q$ ’ (failure) as there is no third event.

$$\text{Probability of an event} = \frac{\text{Number of outcomes favourable to the event}}{\text{Total number of mutually exclusive, exhaustive likely outcomes}}$$

If an event can occur in ‘ $a$ ’ ways and fail to occur in ‘ $b$ ’ ways and these are equally to occur, then the probability of the event occurring,  $\frac{a}{a+b}$  is denoted by  $P$ . Such probabilities are also known as unitary or theoretical or mathematical probability. If  $p$  is the probability of the event happening and  $q$  is the probability of its not happening, then

$$p = \frac{a}{a+b} \text{ and } q = \frac{b}{a+b}$$

$$\text{Hence } p + q = \frac{a}{(a+b)} + \frac{b}{(a+b)} + \frac{a+b}{a+b} = 1$$

Therefore

$$p + q = 1 \text{ or, } 1 - p = q \text{ \& } 1 - q = p$$

Probabilities can be expressed either as ratio, fraction or percentage, such as  $\frac{1}{2}$  or 0.5 or 50%

#### Limitations of Classical Approach

1. This definition is confined to the problems of games of chance only and cannot explain the problem other than the games of chance.
2. We cannot apply this method, when the total number of outcomes cannot be calculated.

3. When the outcomes of a random experiment are not equally likely, this method cannot be applied.
4. It is difficult to subdivide the possible outcomes of a random experiment into mutually exclusive, exhaustive and equally likely in most cases.
5. The definition says that the outcomes are equally likely i.e equiprobable. But before finding the probability how the outcomes can be taken as equiprobable.

**Illustration 1.**

What is the chance of getting a king in a draw from a pack of 52 cards?

**Solution:**

The total number of cases that can happen = 52 (52 cards are there).

Total number of kings are 4 ; hence favourable cases = 4 Therefore probability of drawing a king =  $\frac{4}{52} = \frac{1}{13}$ .

**Illustration 2.**

Two coins are tossed simultaneously. What is the probability of getting a head and a tail?

**Solution:**

The possible combinations of the outcomes when two coins are tossed are HH, HT, TH, TT. The favourable ways are two out of these four possible ways and all these are equally likely to happen.

Hence the probability of getting a head and a tail is  $\frac{2}{4} = \frac{1}{2}$ .

**Illustration 3.**

One card is drawn at random from a well-shuffled pack of 52 cards. What is the probability that it will be (a) a diamond (b) a queen?

**Solution:**

- (a) There are 13 diamond cards in a pack of 52 cards. The number of ways in which a card can be drawn from that pack is 52. The number of cases favourable to the event of getting a diamond is 13.

Hence probability of drawing a diamond

$$\frac{13}{52} = \frac{1}{4}$$

- (b) There are 4 queens in the pack. So the number of cases favourable to the event of getting a queen = 4

$$\text{The required probability} = \frac{4}{52} = \frac{1}{13}$$

**Illustration 4.**

Two cards are drawn from a pack of cards at random. What is the probability that it will be (a) a diamond and a heart (b) a king and a queen (c) two kings?

**Solution:**

- (a) The number of ways of drawing 2 cards from a pack of 52 cards

$$= {}^{52}C_2 = \frac{52 \times 51}{1 \times 2} = 26 \times 51$$

There are 13 each of Diamonds and Hearts in a pack of 52 Cards.

The number of ways of drawing a diamond and a heart =  ${}^{13}C_1 \times {}^{13}C_1 = 13 \times 13$

The required probability =  $\frac{13 \times 13}{26 \times 51} = \frac{13}{102}$

Again there are 4 each of Kings and Queens in a pack of 52 Cards.

(b) The number of ways of drawing a king and a queen =  ${}^4C_1 \times {}^4C_1 = 4 \times 4$

The required probability of drawing a King and a Queen.

$$= \frac{4 \times 4}{26 \times 51} = \frac{6}{663}$$

(c) Two kings can be drawn out of 4 kings in  ${}^4C_2 = \frac{|4}{4-2} \cdot |2}{|2} = \frac{|4}{2} \cdot |2}{|2} = \frac{4}{2} \times \frac{3}{2} = 6$  ways

The probability of drawing 2 kings

$$= \frac{6}{26 \times 51} = \frac{1}{221}$$

### Illustration 5.

A bag contains 7 red, 12 white and 4 green balls. What is the probability that :

(a) 3 balls drawn are all white and

(b) 3 balls drawn are one of each colour ?

#### Solution:

(a) Total number of balls

$$= 7 + 12 + 4 = 23$$

Total number of possible ways of drawing 3 out of 23 balls

$$= {}^{23}C_3 = \frac{|23}{3} \cdot |23-3}{|23-3} = \frac{23 \times 22 \times 21 \times |20}{3 \times 2 \times 1 \times |20} = 1771$$

Number of possible ways of drawing 3 out of 12 white balls

$$= {}^{12}C_3 = \frac{|12}{3} \cdot |12-3}{|12-3} = \frac{12 \times 11 \times 10 \times 9}{3 \times 2 \times 1 \times 10} = 220$$

Therefore, **probability** of drawing 3 white balls =  $\frac{{}^{12}C_3}{{}^{23}C_3} = \frac{220}{1771} = 0.1242 = 0.1242$

(b) Number of possible ways of drawing 1 out of 7 red balls =  ${}^7C_1$

Number of possible ways of drawing 1 out of 12 white balls =  ${}^{12}C_1$

Number of possible ways of drawing 1 out of 4 green balls =  ${}^4C_1$

Therefore the probability of drawing 3 balls of each colours

$$= \frac{{}^7C_1 \times {}^{12}C_1 \times {}^4C_1}{{}^{23}C_3} = \frac{7 \times 12 \times 4}{1771}$$

$$= 0.1897$$

### Relative Frequency Theory of probability

Classical approach is useful for solving problems involving game of chances—throwing dice, coins, etc. but if applied to other types of problems it does not provide answers. For instance, if a man jumps from a height of 300 feet, the probability of his survival will, not be 50%, since survival and death are not equally likely in this case.

Similarly, the prices of shares of a Joint Stock Company have three alternatives i.e. the prices may remain constant or prices may go up or prices may go down. Thus, the classical approach fails to answer questions of these type.

If we toss a coin 20 times, the classical probability suggests that we should have heads ten times. But in practice it may not be so. These empirical approach suggests, that if a coin is tossed a large number of times, say, 1,000 times, we can expect 50% heads and 50% tails. Vor Mises explained, “If the experiment be repeated a large number of times under essentially identical conditions, the limiting value of the ratio of the number of times the event A happens to the total, number of trials of the experiments as the number of trials increases indefinitely, is called the probability of the occurrence of A”.

$$\text{Thus, } P(A) = \lim_{n \rightarrow \infty} \frac{m}{n}$$

The happening of an event is determined on the basis of past experience or on the basis of relative frequency of success in the past. For instance, a machine produces 10% unacceptable articles of the total output. On the basis of such experience or experiments, we may arrive at that (i) the relative frequency obtained on the basis of past experience can be shown to come very close to the classical probability. For example, as said earlier, a coin is tossed for 6 times, we may not get exactly 3 heads and 3 tails. But, the coin is tossed for larger number of times, say 10,000 times, we can expect heads and tails very close to 50% (ii) There are certain laws, according to which the ‘occurrence’ or ‘non-occurrence of the events take place. Posterior probabilities, also called Empirical Probabilities are based on experiences of the past and on experiments conducted. Thus, relative frequency can be termed as a measure of probability and it is calculated on the basis of empirical or statistical findings. For instance if a machine produces 100 articles in the past, 2 particles were found to be defective, then the probability of the defective articles is  $2/100$  or 2%.

### Limitations of Relative Frequency Theory of Probability

1. The experimental conditions may not remain essentially homogeneous and identical in a large number of repetitions of the experiment.
2. The relative frequency —, may not attain a unique value no matter however large  $N$  may be.
3. Probability  $P(A)$  defined can never be obtained in practice. We can only attempt to get a close estimate of  $P(A)$  by making  $N$  sufficiently large.

### Illustration 6.

An urn contains 8 white and 3 red balls. If two balls are drawn at random, find the probability that (a) both are white, (b) both are red and (c) one is of each colour.

#### Solution:

Total number of balls in the urn =  $8 + 3 = 11$

Two balls can be drawn out of 11 balls in  ${}^{11}C_2$  ways.

Exhaustive number of outcomes =  ${}^{11}C_2 = \frac{11 \times 10}{2} = 55$ .

(a) Two white balls to be drawn out of 8 white. It can be done in  $8C_2 = \frac{8 \times 7}{2} = 28$  ways.

The probability that both the balls drawn are white =  $\frac{28}{55}$

(b) Two red balls to be drawn out of 3 red balls. It can be done in  ${}^3C_2 = 3$  ways.

Hence, the probability that both the balls drawn are red =  $\frac{3}{55}$

(c) The number of favourable cases for the event-drawing one white and one red ball is

$${}^8C_1 \times {}^3C_1 = 8 \times 3 = 24.$$

Therefore, the probability of drawing one red and one white ball =  $\frac{24}{55}$

### Illustration 7.

Tickets are numbered from 1 to 100. They are well shuffled and a ticket is drawn at random. What is the probability that the drawn ticket has:

- an even number,
- a number 5 or a multiple of 5,
- a number which is greater than 75,
- a number which is a square?

### Solution:

(a) The total number of exhaustive, mutually exclusive and equally likely outcomes is 100. There are 50 even numbered tickets.

Therefore, number of cases favourable to the event is 50.

Therefore, the probability that the drawn ticket bears an even number =  $\frac{50}{100} = \frac{1}{2}$

(b) Suppose A denotes the event that the drawn ticket has a number 5 or a multiple of 5. These are 5, 10, 15, 20, ... 100. That is 20 such number are possible which are favourable to the event.

$$\text{Therefore, } P(A) = \frac{20}{100} = \frac{1}{5}$$

(c) Between 1 & 100 there are 25 numbers, which are greater than 75. Say B denote the event that the number on the drawn ticket is greater than 75.

$$\text{Therefore, } P(B) = \frac{25}{100} = \frac{1}{4}$$

(d) Between 1 & 100 there are 10 numbers which are perfect squares i.e., 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Let C denotes to event that the number on the drawn ticket is a perfect square.

$$\text{Therefore, } P(C) = \frac{10}{100} = \frac{1}{10}$$

### Illustration 8.

Four cards are drawn one by one from a pack of 52 cards without replacement. What is the probability that they are all of different suits?

### Solution:

There are 52 cards in a full pack which belong to 4 suits @13 each. The first card can be drawn in  ${}_{52}C_1$  ways.

As this is not replaced, there remains 51 cards before drawing the second one.

So the second card can be drawn in  ${}_{51}C_1$  ways.

This way 4 cards can be drawn in  $52_{C_1} \times 51_{C_1} \times 50_{C_1} \times 49_{C_1} = 52 \times 51 \times 50 \times 49$  ways.

Now the drawn cards should be of 4 different suits.

Hence the first card can be of a particular suit in  $(4_{C_1} \times 13_{C_1}) = 4 \times 13 = 52$  ways

Before drawing the second done, there remains 3 different suits. Hence the second card can be of a suit other than

the first are in  $(3_{C_1} \times 13_{C_1}) = 3 \times 13 = 39$

Similarly the third one can be a suit other than the first two in  $(2_{C_1} \times 13_{C_1}) = 2 \times 13 = 26$  ways

The last one can be drawn in  $(1_{C_1} \times 13_{C_1}) = 1 \times 13 = 13$  ways

So the required probability is  $\frac{52 \times 39 \times 26 \times 13}{52 \times 51 \times 50 \times 49} = \frac{2197}{20825}$

**Illustration 9.** A card is drawn at random from a pack of 52 cards. Find probability that the card is a court card.

**Solution:** A = event of getting a court card (Jacks, Queens, Kings)

$$P(A) = \frac{12}{52} = \frac{3}{13}$$

**Illustration 10.** Two cards are thrown at once. What is Prob. Of getting doublets.

**Solution:** A = event of getting doublets.

$$P(A) = \frac{6}{36} = \frac{1}{6}, \quad \text{No. of Fav cases} = (1,1) (2,2) \dots \dots \dots (6,6)$$

**Illustration 11.** A card is drawn from a pack of 52 cards find odds in favour of getting a king card.

**Solution:** A = event of getting a king card.

$$P(A) = \frac{4}{52} = \frac{1}{13}, \quad P(A) = 1 - \frac{1}{13} = \frac{12}{13}$$

Odds in favour of getting a king card

$$= \frac{P(A)}{P(A')} = \frac{1}{13} \times \frac{13}{12} = \frac{1}{12}$$

**Illustration 12.** A number is selected at random from a set of 25 natural nos. find Prob. That it would be divisible by 4 or 7.

**Solution:** A = event of getting a number

Which is divisible by 4 or 7.

$$= (4,8,12,16,20,24,7,14,21)$$

$$P(A) = \frac{9}{25} = \frac{n(A)}{n(S)}$$

# Primary applications of Probability Theorems

## 7.2

### Theorems of Probability

We have studied what probability is and how it can be measured. We dealt with simple problems. Now we shall consider some of the laws of probability to tackle complex situation. There are two important theorems, viz., (1) the Addition Theorem and (2) the Multiplication Theorem.

#### Addition Theorem

The simplest and most important rule used in the calculation is the addition rule, it states, “If two events are **mutually exclusive**, then the probability of the occurrence of either  $A$  or  $B$  is the sum of the probabilities of  $A$  and  $B$ . Thus,

$P(A \text{ or } B) = P(A + B) = P(A) + P(B)$ , when  $A$  and  $B$  are mutually exclusive.

#### Illustration 13.

A bag contains 4 white, 3 black and 5 red balls. What is the probability of getting a white or a red ball at random in a single random draw?

#### Solution:

The probability of getting a white ball =  $\frac{4}{12}$

The probability of getting a red ball =  $\frac{5}{12}$

The probability of getting a white or a red =  $\frac{4}{12} + \frac{5}{12} = \frac{9}{12} = \frac{9}{12} \times 100 = 75\%$

#### When events are **not mutually exclusive**

The addition theorem studied above is not applicable when the events are not mutually exclusive. In such cases where the events are not mutually exclusive, the probability is :

$P(\text{At least one of } A \text{ or } B) = P(A + B) = P(A) + P(B) - P(A \text{ and } B)$ , when  $A$  and  $B$  are not mutually exclusive

#### Illustration 14.

Two students  $X$  and  $Y$  work independently on a problem. The probability that  $A$  will solve it is  $\frac{3}{4}$  and the probability that  $Y$  will solve it is  $\frac{2}{3}$ . What is the probability that the problem will be solved?

#### Solution:

The problem will be solves if at least one of  $X$  and  $Y$  solve it

$P(\text{A least one of } A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

The probability that X will solve the problem is  $\frac{3}{4} = P(A)$

The probability that Y will solve the problem is  $\frac{2}{3} = P(B)$

The events are not mutually exclusive as both of them may solve the problem.

$$\begin{aligned} \text{Therefore, the probability} &= \frac{3}{4} + \frac{2}{3} - \left( \frac{3}{4} \times \frac{2}{3} \right) \\ &= \frac{17}{12} - \frac{6}{12} = \frac{11}{12} \end{aligned}$$

**Alternatively:**

The probability that X will solve it and Y fail to solve it =  $\frac{3}{4} \times \frac{1}{3} = \frac{3}{12}$

$$\therefore \text{Probability that the problem will be solved} = \frac{2}{3} + \frac{3}{12} = \frac{11}{12}$$

**Alternatively**

The probability that X will fail to solve and will Y solve it

$$= \frac{1}{4} \times \frac{2}{3} = \frac{2}{12}$$

$$\therefore \text{Probability that the problem will be solved} = \frac{3}{4} + \frac{2}{12} = \frac{9+2}{12} = \frac{11}{12}$$

**Alternatively**

The probability that neither X nor Y will solve it  $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

Hence, the probability that the problem will be solved

$$= 1 - \frac{1}{12} = \frac{11}{12}$$

### Illustration 15.

Three events A, B & C are mutually exclusive, equally likely & exhaustive. What is the Prob. of complementary of event A.

**Solution:**

A, B & C are mutually exclusive events.

Then  $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

They are exhaustive events.

Then  $P(A) + P(B) + P(C) = 1$

They are equally likely events.

$P(A) = P(B) = P(C)$

Then  $P(A) + P(A) + P(A) = 1$

Or,  $3P(A) = 1$  or  $P(A) = \frac{1}{3}$

$P(A) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$

**Illustration 16.** A card is drawn from a pack of 52 cards. What is Prob. that it is a heart or a king.

**Solution:** A = event of drawing a heart.

B = event of drawing a king.

$$P(A) = \frac{13}{52}, P(B) = \frac{4}{52}, P(A \cap B) = \frac{1}{52}$$

Prob. that it is a heart or a king

$$\begin{aligned} &= P(A \cup B) = P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} \end{aligned}$$

**Illustration 17.** A coin and a dice are thrown. What is prob. of getting a head or an even number.

**Solution:** P(a head) =  $\frac{1}{2}$ , P(an even no. of dice) =  $\frac{3}{6} = \frac{1}{2}$

P(a head or an even no.)

= P(a head) + P(an even no.) - P(a head and an even no.)

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$$

(P(A ∩ B) = P(A) × P(B))

**Illustration 18.** Two cards are drawn from a pack of 52 cards with replacement (2nd card is drawn after replacing 1st card). Find prob. that (a) both are king. (b) 1st is king & 2nd is queen. (c) one is king & other is queen.

**Solution:** a) P(both cards are king) =  $\frac{4}{52} \times \frac{4}{52}$

b) P(1st is king & 2nd is queen) =  $\frac{4}{52} \times \frac{4}{52}$

c) P(One is king & other is queen)

= P(1st is king & 2nd is queen) + P(1st is queen & 2nd is king)

$$= \frac{4}{52} \times \frac{4}{52} + \frac{4}{52} \times \frac{4}{52}$$

**Illustration 19.** Two cards are drawn from a pack of 52 cards without replacement. (2nd card is drawn without replacing 1st card). What is prob. that (a) both cards are king. (b) 1st card is king & 2nd card is queen)

**Solution:** (a) P(both cards are king) =  $\frac{4}{52} \times \frac{3}{51}$

(b) P(1st card is king & 2nd card is queen) =  $\frac{4}{52} \times \frac{4}{51}$

**Illustration 20.** A speaks truth is 70% and B is 80%. In what percentage of cases they likely to contradict each other in speaking truth.

**Solution:** P(A) = .7, P(B) = .8, P(A') = .3, P(B') = .2

P(A & B will contradictory)

= P(A) × P(B') + P(B) × P(A')

$$= .7 \times .2 + .8 \times .3 = .14 + .24$$

$$= .38$$

**Multiplication Theorem**

When it is desired to estimate the chances of the happening of successive events, the separate probabilities of these successive events are multiplied. If two events A and B are **independent**, then the probability that both will occur is equal to the product of the respective probabilities. We find the probability of the happening of two or more events in succession.

$P(A \text{ and } B) = P(AB) = P(A) \times P(B)$  when A and B are independent

**Illustration 21.**

In two tosses of a fair coin, what are the chances of head in both ?

**Solution:**

Probability of head in first toss =  $1/2$

Probability of head in the second toss =  $1/2$

Probability of head in both tosses =  $1/2 \times 1/2 = 1/4$

**Illustration 22.**

The probability that X and Y will be alive ten years hence is 0.5 and 0.8 respectively. What is the probability that both of them will be alive ten years hence?

**Solution:**

Probability of X being alive ten years hence = 0.5

Probability of Y being alive ten years hence = 0.8

Probability of X and Y both being alive ten years hence =  $0.5 \times 0.8 = 0.4$

**When events are dependent:**

If the events are dependent, the probability is conditional. Two events A and B are dependent ; B occurs only when A is known to have occurred.

$P(B/A)$  means the probability of occurrence of B given that A has occurred.

$$P(B/A) = \frac{P(AB)}{P(A)}; P\left(\frac{A}{B}\right) = \left(\frac{P(AB)}{P(B)}\right)$$

$$P(A \& B) = P(AB) = P(A) \cdot P\left(\frac{B}{A}\right) = P(B) \cdot P\left(\frac{A}{B}\right) \text{ when A and B are not independent}$$

**Illustration 23.**

A man want to marry a girl having qualities: White complexion the probability of getting such girl is 1 in 20. Professionally qualified - the probability of getting is 1 in 50. Westernised style - the probability is 1 in 100.

Find out the probability of his getting married to such a girl, who has all the three qualities.

**Solution:**

The probability of a girl with white complexion =  $\frac{1}{20}$  or 0.05. The probability of a girl with professional qualification

$= \frac{1}{50}$  or 0.02. The probability of a girl with westernised style  $= \frac{1}{100}$  or 0.01. Since the events are independent, the probability of getting married to a girl having all three qualities  $= \frac{1}{20} \times \frac{1}{50} \times \frac{1}{100} = 0.05 \times 0.02 \times 0.01 = 0.00001$

#### Illustration 24.

A university has to select an examiner from a list of 50 persons, 20 of them are women and 30 men. 10 of them know Hindi and 40 do not. 15 of them are teachers and the remaining 35 are not. What is the probability of the University selecting a Hindi-knowing woman teacher?

#### Solution:

Probability of selecting a woman  $= \frac{20}{50}$

Probability of selecting a teacher  $= \frac{15}{50}$

Probability of selecting a Hindi-knowing candidate  $= \frac{10}{50}$

Since the events are independent the probability of the University selecting a Hindi-knowing woman teacher is:

$$\frac{20}{50} \times \frac{15}{50} \times \frac{10}{50} = \frac{3}{125} \text{ or } 0.024.$$

#### Illustration 25.

A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Determine the probability that it is :

(i) Red (ii) white, (iii) Blue, (iv) Not Red and (v) Red or White.

#### Solution:

There are 6 Red, 4 White and 5 Blue balls i.e.  $6 + 4 + 5 = 15$  balls in the box.

(i) Probability of Red  $= \frac{6}{15}$  or 0.40

(ii) Probability of white  $= \frac{4}{15}$  or 0.267

(iii) Probability of Blue  $= \frac{5}{15}$  or 0.333

(iv) Probability of not Red  $= \frac{9}{15}$  or 0.60

(v) Probability of Red or White  $= \frac{6}{15} + \frac{4}{15} = \frac{10}{15} = 0.667$

### BAYES' THEOREM

This theorem is associated with the name of Reverend **Thomas Bayes**. It is also known as the inverse probability. Probabilities can be revised when new information pertaining to a random experiment is obtained. One of the important applications of the conditional probability is in the computation of unknown probabilities, on the basis of the information supplied by the experiment or past records. That is, the applications of the results of probability theory involves estimating unknown probabilities and making decisions on the basis of new sample information.

This concept is referred to as Bayes' Theorem. Quite often the businessman has the extra information on a particular event, either through a personal belief or from the past history of the events. Revision of probability arises from a need to make better use of experimental information. Probabilities assigned on the basis of personal experience, before observing the outcomes of the experiment are called prior probabilities. For example, probabilities assigned to past sales records, to past number of defectives produced by a machine, are examples of prior probabilities. When the probabilities are revised with the use of Bayes' rule, they are called posterior probabilities. Bayes' theorem is useful in solving practical business problems in the light of additional information. Thus popularity of the theorem has been mainly because of its usefulness in revising a set of old probability (Prior Probability) in the light of additional information made available and to derive a set of new probability (i.e. Posterior Probability)

**Bayes' Theorem:** An event A can occur only if one of the mutually exclusive and exhaustive set of events  $B_1, B_2, \dots, B_n$  occurs. Suppose that the unconditional probabilities  $P(B_1), P(B_2), \dots, P(B_n)$  and the conditional probabilities  $P(A/B_1), P(A/B_2), \dots, P(A/B_n)$ .

are known. Then the conditional probability  $P(B_i/A)$  of a specific event  $B_i$ , when A is stated to have actually occurred, is given by

$$P(B_i / A) = \frac{P(B_i) \cdot P(A / B_i)}{\sum_{i=1}^n P(B_i) \cdot P(A / B_i)}$$

This is known as Bayes' Theorem.

The following examples illustrate the application of Bayes' Theorem.

**Illustration 26.**

You note that your officer is happy on 60% of your calls, so you assign a probability of his being happy on your visit as 0.6 or 6/10. You have noticed also that if he is happy, he accedes to your request with a probability of 0.4 or 4/10 whereas if he is not happy, he accedes to the request with a probability of 0.1 or  $\frac{1}{10}$ . You call one day, and he accedes to your request. What is the probability of his being happy ?

**Solution:**

Let- H be the event that the officer is happy and  $\bar{H}$  be the event that the officer is not happy

$$P(H) = \frac{6}{10} \text{ (Given) } P(\bar{H}) = 1 - P(H) = 1 - \frac{6}{10} = \frac{4}{10}$$

Let A be the event that he accedes to your request

$$P(A / H) = \frac{4}{10}, \quad P(A / \bar{H}) = \frac{1}{10} \text{ (As per given in formation)}$$

To find P(H/A), we use Bayes' Theorem as follows–

$$P(H / A) = \frac{P(H) \times P\left(\frac{A}{H}\right)}{P(H) \times P\left(\frac{A}{H}\right) + P(\bar{H}) \times P\left(\frac{A}{\bar{H}}\right)} = \frac{\frac{6}{10} \times \frac{4}{10}}{\frac{6}{10} \cdot \frac{4}{10} + \frac{4}{10} \cdot \frac{1}{10}}$$

$$= \frac{\frac{24}{100}}{\frac{24}{100} + \frac{4}{100}} = \frac{24}{28} = \frac{6}{7} = 0.857$$

**Illustration 27.**

A company has two plants to manufacture scooters. Plant I manufactures 80% of the scooters and plant II manufactures 20%. At Plant I, 85 out of 100 scooters are rated standard quality or better. At Plant II, only 65 out of 100 scooters are rated standard quality or better. What is the probability that the scooter selected at random came from Plant I if it is known that the scooter is of standard quality?

What is the probability that the scooter came from Plant II if it is known that the scooter is of standard quality.

**Solution:**

Let  $A_1$  be the event of drawing a scooter produced by Plant I and  $A_2$  be the event of drawing a scooter produced by Plant II. B be the event of drawing a standard quality scooter produced by either Plant I or Plant II

Then, from the first information :

$$P(A_1) = \frac{80}{100} = 80\% = 0.80 \text{ and } P(A_2) = \frac{20}{100} = 20\% = 0.20$$

From the additional information :

$$P\left(\frac{B}{A_1}\right) = \frac{85}{100} = 85\% \text{ and } P\left(\frac{B}{A_2}\right) = \frac{65}{100} = 65\%$$

The required values are computed in the following table :

Event	Prior Probability	Conditional Probability	Joint Probability	Posterior Probability (Revised) (5) $[4 \div P(B)]$
(1)	(2)	(3)	(4) = (2) $\times$ (3)	(5) = (4) $\div$ P(B)
$A_1$	0.80	0.85	0.68	$\frac{0.68}{0.81} = \frac{68}{81}$
$A_2$	0.20	0.65	0.13	$\frac{0.13}{0.81} = \frac{13}{81}$
Total	1	–	$P(B) = 0.81$	1

The above calculation can be verified as follows :

If 1,000 scooters were produced by the two plants in a particular week, the number of scooters produced by Plant I & Plant II are respectively :

$$1,000 \times 80\% = 800 \text{ scooters}$$

$$1,000 \times 20\% = 200 \text{ scooters}$$

The number of standard quality scooters produced by Plant I :

$$800 \times 85/100 = 680 \text{ scooters}$$

The number of standard quality scooters produced by Plant II :

$$200 \times 65/100 = 130 \text{ Scooters.}$$

The probability that a standard quality scooter was produced by Plant I is :

$$= \frac{680}{680+130} = \frac{680}{810} = \frac{68}{81}$$

The probability that a standard quality scooter was produced by Plant II is :

$$= \frac{130}{680+130} = \frac{130}{810} = \frac{13}{81}$$

From the first information we may say that the standard scooter is drawn from Plant I since  $P(A_1) = 80\%$  which is greater than  $P(A_2) = 20\%$ ,

From the additional information i.e. at Plant I, 85 out of 100 and at Plant II 65 out of 100 are rated standard quality, we can give better answer, Thus we may conclude that the standard quality of scooter is more likely drawn from the output by Plant I.

The same process i.e. revision can be repeated if more information is made available. Thus it is a good theorem in improving the quality of probability in decision making under uncertainty.

### Illustration 28.

Box I contains three defective and seven non-defective balls, and Box II contains one defective and nine non-defective balls. We select a box at random and then draw one ball at random from the box.

- What is the probability of drawing a non-defective ball?
- What is the probability of drawing a defective ball?
- What is the probability that box I was chosen, given a defective ball is drawn?

#### Solution:

As no specific information about the look of the boxes are provided, it is assumed that they are identical in appearance.

So  $P(B_1)$  or Probability that Box I is chosen =  $\frac{1}{2}$

$P(B_2)$  or Probability that Box II is chosen =  $\frac{1}{2}$

Let  $P(D)$  - Probability that a defective Ball is drawn and  $P(ND)$  = Probability that a non-defective Ball is drawn

Joint Probability

(i) Probability of selecting Box I and getting a defective ball from it =  $\frac{1}{2} \times \frac{3}{10} = \frac{3}{20}$

(ii) Probability of selecting Box II and getting a defective ball from it =  $\frac{1}{2} \times \frac{1}{10} = \frac{1}{20}$

(iii) Probability of selecting Box I and getting a non defective ball from it =  $\frac{1}{2} \times \frac{7}{10} = \frac{7}{20}$

(iv) Probability of selecting Box II and getting a non defective ball from it =  $\frac{1}{2} \times \frac{9}{10} = \frac{9}{20}$

(a)  $P(ND) = P(\text{Box I and non-defective}) + P(\text{Box II and non-defective}) = P[\text{Case (iii)}] + P[\text{Case (iv)}]$

$$= \left(\frac{1}{2} \times \frac{7}{10}\right) + \left(\frac{1}{2} \times \frac{9}{10}\right) = \frac{16}{20}$$

(b)  $P(D) = P(\text{Box I and defective}) + P(\text{Box II and defective})$

$$= \left(\frac{1}{2} \times \frac{3}{10}\right) + \left(\frac{1}{2} \times \frac{1}{10}\right) = \frac{4}{20}$$

(c) Bayes' Theorem :

$$P(B_1 / D) = \frac{P(B_1 \text{ and } D)}{P(D)} = \frac{3/20}{4/20} = \frac{3}{4}$$

$P(B_1)$  and  $P(B_2)$  are called prior probabilities and  $P\left(\frac{B_1}{D}\right)$  and  $P\left(\frac{B_2}{D}\right)$  are called posterior probabilities. The above information is summarised in the following table :

Event	Prior Probability	Conditional Probability	Joint Probability	Posterior Probability
(1)	(2)	(3)	(4) = (2) × (3)	(5) = (4) ÷ P(D)
B <sub>1</sub>	$\frac{1}{2}$	3/10	3/20	3/4
B <sub>2</sub>	$\frac{1}{2}$	1/10	1/20	1/4
Total	1	–	$P(D) = 4/20$	1

## ODDS

We must know the concept of odds. The word “Odds” is frequently used in statistics. Odds relate the chances in favour of an event to the chances against it. For instance, the odds are 2 : 1 that A will get a job, means that there are 2 chances that he will get the job and 1 chance against his getting the job. This can also be converted into probability as getting the job =  $\frac{2}{3}$ . Therefore, if the odds are  $a : b$  in favour of an event then  $P(A) = \frac{a}{a+b}$ . Further, it may be noted that the odds are  $a : b$  in favour of an event is the same as to say that the odds are  $b : a$  against the event.

If the probability of an event is  $p$ , then the odds in favour of its occurrence are  $p$  to  $(1-p)$  and the odds against its occurrence are  $1-p$  to  $p$ .

### Illustration 29.

Suppose it is 11 to 5 against a person A who is now 38 years of age living till he is 73 and 5 to 3 against B who is 43 living till he is 78. Find the chance that at least one of these persons will be alive 35 years hence.

**Solution:**

The probability that A will die within 35 years =  $\frac{11}{11+5} = \frac{11}{16}$

The probability that B will die within 35 years =  $\frac{5}{5+3} = \frac{5}{8}$

The probability that both of them will die within 35 years

$$= \frac{11}{16} \times \frac{5}{8} = \frac{55}{128}$$

The probability that both of them will not die i.e. atleast one of them will be alive

$$= 1 - \frac{55}{128} = \frac{73}{128} \text{ or } \frac{73}{128} \times 100 = 57\%$$

**Illustration 30.**

Two cards are drawn at random one by one without replacement from a well-shuffled pack of 52 cards. What is the probability that :

- both are aces,
- both are red,
- at least one is an ace?

**Solution:**

- (a) Let  $A$  indicate the event of drawing an ace. So drawing of aces on both occasions can be denoted as  $AA$ .

$$P(AA) = P(A) \times P\left(\frac{A}{A}\right)$$

$P(A)$  : Probability of drawing of an ace first

$P\left(\frac{A}{A}\right)$  : Probability of an ace at the second draw, given that the first was an ace.

Therefore,  $P(A) = \frac{4}{52}$  [As there are 4 aces in a pack of 52 cards]

$P\left(\frac{A}{A}\right) = \frac{3}{51}$  [After drawing an ace for the first time there remains 51 cards in the pack of which 3 are aces.]

$$P(AA) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

- (b) Let  $R$  indicate the event of drawing a red card. So the event of drawing red cards on both occasions can be denoted as  $RR$

$$P(RR) = P(R) \times P\left(\frac{R}{R}\right)$$

$= \frac{26}{52} \times \frac{25}{51}$  [As there are 26 red cards in a pack of 52 cards and also there remains 51 cards after the first drawing]

of which 25 are red]

$$= \frac{25}{102}$$

- (c) Let  $E$  indicate the event of drawing not an ace. Then the probability that no aces are drawn on both occasions is indicated as  $P(EE)$ . Probability of not drawing an ace :

$$\begin{aligned} P(EE) &= P(E) \times P\left(\frac{E}{E}\right) \\ &= \frac{48}{52} \times \frac{47}{51} = \frac{188}{221} \end{aligned}$$

Therefore, probability of drawing at least one ace

$$= 1 - \frac{188}{221} = \frac{33}{221}$$

### Illustration 31.

The odds in favour of a certain event are 2 to 5 and the odds against another event independent of the former are 5 to 6. Find the chance that at least one of the events will happen.

#### Solution:

Odds in favour of the 1st event ( $A$ , let) = 2:5

$$\text{So, } P(A) = \frac{2}{2+5} = \frac{2}{7}$$

Odds against the 2nd event ( $B$ , let) = 5:6

$$\text{So, } P(A) = \frac{6}{5+6} = \frac{6}{11}$$

Probability that at least are of the events will happen =  $P(A+B) = P(A) + P(B) - P(AB)$

$$\text{or, } P(A+B) = \frac{2}{7} + \frac{6}{11} - \frac{2}{7} \times \frac{6}{11} \quad [\text{As } A \text{ \& } B \text{ are independent, } P(AB) = P(A) \times P(B)]$$

$$\text{or, } P(A+B) = \frac{2}{7} + \frac{6}{11} - \frac{12}{77}$$

$$\text{or, } P(A+B) = \frac{22 + 42 - 12}{77}$$

$$\text{or, } P(A+B) = \frac{52}{77}$$

#### Alternatively:

Probability that the first event does not happen

$$= 1 - \frac{2}{7} = \frac{5}{7}$$

Probability that the second event does not happen

$$= 1 - \frac{6}{11} = \frac{5}{11}$$

The chance that both do not happen

$$= \frac{5}{7} \times \frac{5}{11} = \frac{25}{77} \text{ [As the events are independent]}$$

$$\text{The chance that one at least will happen} = 1 - \frac{25}{77} = \frac{52}{77}$$

### Illustration 32.

What is the chance that a leap year, selected at random will contain 53 Sundays?

#### Solution:

As a leap year consists of 366 days, it contains 52 complete weeks (i.e 52 Sunday for sure) and two more days.

The two consecutive days may appear in either of the following combinations :

- (a) Monday and Tuesday
- (b) Tuesday and Wednesday
- (c) Wednesday and Thursday
- (d) Thursday and Friday
- (e) Friday and Saturday
- (f) Saturday and Sunday, and
- (g) Sunday and Monday

If (f) or (g) occur, then the year consists of  $52 + 1 = 53$  Sundays.

Therefore the number of favourable cases = 2

Total number of cases = 7

The required probability =  $\frac{2}{7}$

### Illustration 33.

A problem in statistics is given to three students A, B, C whose chances of solving it are  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$  respectively. What is the probability that the problem will be solved?

#### Solution:

The probability that A fails to solve the problem =  $1 - \frac{1}{2} = \frac{1}{2}$

The probability that B fails to solve the problem  $= 1 - \frac{1}{3} = \frac{2}{3}$

The probability that C fails to solve the problem  $= 1 - \frac{1}{4} = \frac{3}{4}$

The probability that the problem is not solved by A, B and C  $= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$  (As the events are independent)

Therefore, the probability that the problem is solved  $= 1 - \frac{1}{4} = \frac{3}{4}$

### Illustration 34.

An ordinary die is tossed twice and the difference between the number of spots turned up is noted. Find the probability of a difference of 3.

#### Solution:

The sample space consists of 36 values.

The event space has the following 6 cases : (1, 4), (2, 5), (3, 6), (4, 1), (5, 2), (6, 3)

The required probability  $= \frac{6}{36}$

### Illustration 35.

From a pack of 52 cards, two cards are drawn at random ; find the chance that one is a knave and the other a queen.

#### Solution:

No. of elements in the Sample space  $= {}^{52}C_2$

No. of elements in the Event space  $= {}^4C_1 \times {}^4C_1$

(as there are 4 queens and 4 knaves in the pack)

Required Probability  $= \frac{{}^4C_1 \times {}^4C_1}{{}^{52}C_2} = \frac{8}{663}$

### Illustration 36.

A bag contains 7 red balls and 5 white balls. 4 balls are drawn at random. What is the probability that (i) all of them are red; (ii) two of them are red and two white?

#### Solution:

(i) Favourable cases  ${}^7C_4$ , Exhaustive cases  ${}^{12}C_4$

$$\text{Probability} = \frac{{}^7C_4}{{}^{12}C_4} = \frac{35}{495} = \frac{7}{99}$$

(ii) Favourable cases  $= {}^7C_2 \times {}^5C_2$

Exhaustive cases  $= {}^{12}C_4$

$$\text{Probability} = \frac{{}^7C_2 \times {}^5C_2}{{}^{12}C_4} = \frac{21 \times 10}{495} = \frac{14}{33}$$

**Illustration 37.**

A petrol pump proprietor sells on an average ₹ 80,000 worth of petrol on rainy days and an average of ₹ 95,000 on clear days. Statistics from the Metereological Department show that the probability is 0.76 for clear weather and 0.24 for rainy weather on coming Monday. Find the expected value of petrol sale on coming Monday.

**Solution:**

$$X_1 = ₹ 80,000; P_1 = 0.24$$

$$X_2 = ₹ 95,000 P_2 = 0.76$$

$$\begin{aligned} \text{Mathematical expectation} &= \Sigma PX = P_1 X_1 + P_2 X_2 \\ &= 0.24 X 80,000 + 0.76 X 95,000 \\ &= 19,200 + 72,200 = ₹ 91,400. \end{aligned}$$

The expected value of petrol sale on coming Monday = ₹ 91,400

**Illustration 38.**

A bag contains 6 white and 9 black balls. Two drawings of 4 balls are made such that (a) the Balls are replaced before the second trial (b) the balls are not replaced before the second trial. Find the probability that the first drawing will give 4 white and the second 4 black balls in each case.

**Solution:**

- (a) When the balls are replaced before the second trial, the number of ways in which 4 balls may be drawn is  ${}^{15}C_4$  in each drawing

$$\text{The number of ways in which 4 white balls can be drawn} = {}^6C_4$$

$$\text{The number of ways in which 4 black balls can be drawn} = {}^9C_4$$

Therefore, the probability of drawing 4 white balls at first trial

$$= \frac{{}^6C_4}{{}^{15}C_4} = \frac{1}{91}$$

The Second trial of drawing 4 black balls.

$$= \frac{{}^9C_4}{{}^{15}C_4} = \frac{9 \times 8 \times 7 \times 6}{4!} \times \frac{4!}{15 \times 14 \times 13 \times 12} = \frac{6}{65}$$

$$\text{Therefore the chance of the Compound event} = \frac{1}{91} \times \frac{6}{65} = \frac{6}{5915}$$

- (b) When the balls are not replaced :

At the first trial, 4 balls may be drawn in  ${}^{15}C_4$  ways and 4 white balls may be drawn in  ${}^6C_4$  ways.

$$\text{Therefore the chance of 4 white balls at first trial} = \frac{{}^6C_4}{{}^{15}C_4} = \frac{1}{91} \text{ (as above)}$$

When 4 white balls have been drawn and removed, the bag contains 2 white and 9 black balls.

Therefore at the second trial, 4 balls can be drawn in  ${}^{11}C_4$  ways and 4 black balls can be drawn in  ${}^9C_4$  ways So, the chance of 4 black balls at the second trial

$$= \frac{{}^9C_4}{{}^{11}C_4}$$

$$= \frac{9 \times 8 \times 7 \times 6}{4!} \times \frac{4!}{11 \times 10 \times 9 \times 8} = \frac{21}{55}$$

Therefore the chance of the compound event =  $\frac{1}{91} \times \frac{21}{55} = \frac{3}{715}$

### Illustration 39.

A salesman is known to sell a product in 3 out of 5 attempts while another salesman in 2 out of 5 attempts. Find the probability that (i) No sale will be effected when they both try to sell the product and (ii) Either of them will succeed in selling the product.

#### Solution:

Let the two salesmen be A and B.

$P(A)$  = The probability that the salesman A is able to sell the product =  $\frac{3}{5}$

$P(B)$  = The probability that the salesman B is able to sell the product =  $\frac{2}{5}$

(i) probability that no sale will be effected =  $\left(1 - \frac{3}{5}\right)\left(1 - \frac{2}{5}\right) = \frac{6}{25}$

(ii) probability that either of them will succeed in selling the product =  $P(A + B) = P(A) + P(B) - P(AB)$

$$= \frac{3}{5} + \frac{2}{5} - \frac{3}{5} \times \frac{2}{5} = \frac{19}{25}$$

### Illustration 40.

A class consists of 100 students, 25 of them are girls and 75 boys, 20 of them are rich and remaining poor, 40 of them are fair complexioned. What is the probability of selecting a fair complexioned rich girl?

#### Solution:

Probability of selecting a fair complexioned student =  $\frac{40}{100} = \frac{2}{5}$

Probability of selecting a rich student =  $\frac{20}{100} = \frac{1}{5}$

Probability of selecting a girl =  $\frac{25}{100} = \frac{1}{4}$

Since the events are independent, by multiplication rule of probability, the

probability of selecting a fair complexioned rich girl =  $\frac{2}{5} \times \frac{1}{5} \times \frac{1}{4} = \frac{2}{100} = 0.02$

**Illustration 41.**

Three groups of workers contain 3 men and one woman, 2 man and 2 women, and 1 man and 3 women respectively. One worker is selected at random from each group. What is the probability that the group selected consists of 1 man and 2 women?

**Solution:**

There are three possibilities :

- (i) Man is selected from the first group and women from second and third group
- (ii) Man is selected from the second group and women from first and third groups
- (iii) Man is selected from the third group and women from first and second groups.

∴ the probability of selecting a group of one man & two woman

$$= \left(\frac{3}{4} \times \frac{2}{4} \times \frac{3}{4}\right) + \left(\frac{2}{4} \times \frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4} \times \frac{2}{4}\right)$$

$$= \frac{18}{64} + \frac{6}{64} + \frac{2}{64} = \frac{13}{32}$$

**Exercise:****Theoretical Questions**⊙ **Multiple Choice Questions (MCQ)**

1. Probability theory is often referred to as
  - (a) Science of prediction
  - (b) Science of uncertainty
  - (c) Science of chance
  - (d) Science of decision making
  
2. In general probability is
  - (a) A numerical value between 0 and 1, exclusive, describing the absolute possibility an event will occur
  - (b) A numerical value between 0 and 1, inclusive, describing the absolute possibility an event will occur
  - (c) A numerical value between 0 and 1, exclusive, describing the relative possibility an event will occur
  - (d) A numerical value between 0 and 1, inclusive, describing the relative possibility an event will occur
  
3. The probability of two events A and B are 0.05 and 0.95 respectively. We can infer that
  - (a) Event A is more probable to happen
  - (b) Event B is more improbable to happen
  - (c) Event B is more probable to happen
  - (d) Event A & B are sure to happen
  
4. “Sun will disappear from blue sky today forever”. With our available information & belief which one of the following value is most appropriate as probability to this event?
  - (a) 0.2
  - (b) 0.8
  - (c) 1
  - (d) 0
  
5. For an event Odds in favour are “five to two” This means that
  - (a) In a total of seven trials the event will occur five times
  - (b) In a total of seven trials the event will occur two times
  - (c) In a total of five trials the event will occur two times
  - (d) In a total of seven trials the event will not occur five times

6. Classical probability is based on the assumption that
- The outcomes of an experiment are already known
  - The outcomes of an experiment are equally likely
  - The probability of all outcomes in an experiment is always 1
  - None of the above
7. If an experiment has a set of events that includes every possible outcomes, then the set is called
- Mutually Exclusive set
  - Mutually Exhaustive set
  - Collectively Exhaustive set
  - Exhaustive & Exclusive set
8. Classical probability is
- Probability of an event =  $\frac{\text{Number of favourable outcome}}{\text{Total number of possible outcome}}$
  - Probability of an event =  $\frac{\text{Number of favourable outcome}}{\text{Total number of trails}}$
  - Probability of an event =  $\frac{\text{Number of possible outcome}}{\text{Total number of trails}}$
  - Probability of an event =  $\frac{\text{Number of trials}}{\text{Number of possible outcomes}}$
9. Addition rule for mutually exclusive events A & B is
- $P(A \text{ or } B) = P(A) + P(B)$
  - $P(A \text{ or } B) = P(A + B)$
  - $P(A \text{ or } B) = P(A) + P(B) - P(AB)$
  - $P(A \text{ or } B) = P(A + B - AB)$
10. If two unbiased coins are tossed once, the probability of getting both the heads is
- 0.25
  - 0.50
  - 0.75
  - 1

11. The probability that a leap year selected at random contain 53 Sundays is
- (a) 0.143
  - (b) 1
  - (c) 0.286
  - (d) 0.48
12. A lot contains 10 items of which 3 are defective. Three items are chosen from the lot at random one after another without replacement. The probability that all the three are defective is
- (a) 0.008
  - (b) 0.992
  - (c) 0.067
  - (d) 0.05
13. If  $P(A) = 0.3, P(B) = 0.2$  and  $P(C) = 0.1$ , then assuming A,B and C are independent events, the probability of occurrence of at least one of the three events is
- (a) 0.7
  - (b) 0.8
  - (c) 0.006
  - (d) 0.496
14. In IPL Kolkata Knight Riders plays 70% of their games at night (8 O'clock slot) and 30% during the day (4 O'clock slot). The team wins 50% of their night games and 90% of their day games. According to today's newspaper they own yesterday. The probability that the game was played at night is
- (a) 0.4667
  - (b) 0.5645
  - (c) 0.35
  - (d) 0.5
15. When two events happen simultaneously which of the following is true?
- (a) The outcome of the first event always have an effect on the outcome of the second event
  - (b) The outcome of the first event may or may not have an effect on the outcome of the second event
  - (c) The outcome of the first event does not not have any effect on the outcome of the second event
  - (d) The outcome of the first event have always a 50% effect on the outcome of the second event

16. A survey by Air travelers' association revealed that 60% of its member made airline reservations last year. Two members are selected at random. The probability that both the members made airline reservations last year is
- 0.60
  - 0.40
  - 0.36
  - 0.16
17. If an unbiased coin is tossed once, then the two events head and tail are
- Mutually exclusive
  - Exhaustive
  - Equally likely
  - All these
18. Sum of probability of an event A and its complement is \_\_\_
- 1
  - 0
  - $\frac{1}{2}$
  - $-\frac{1}{2}$
19. If p: q are the odds in favour of an event, then the probability of that event is
- $p/q$
  - $p/(p + q)$
  - $q/(p + q)$
  - None of these
20. A bag contains 30 balls numbered from 1 to 30. One ball is drawn at random. The probability that the number of the drawn ball will be multiple of 3 or 7 is
- $7/15$
  - $13/30$
  - $1/2$
  - None of these

21. Three coins are tossed together. The probability of getting exactly two heads is
- (a)  $\frac{5}{8}$
  - (b)  $\frac{3}{8}$
  - (c)  $\frac{1}{8}$
  - (d) None
22. Two dice are thrown together. The probability of the event that the sum of numbers shown is greater than 5 is
- (a)  $\frac{13}{18}$
  - (b)  $\frac{15}{18}$
  - (c) 1
  - (d) None
23. Probability of throwing an even number with an ordinary six faced die is
- (a)  $\frac{1}{2}$
  - (b) 1
  - (c) 0
  - (d)  $-\frac{1}{2}$
24. 4 coins are tossed. The probability that there are 2 heads is
- (a)  $\frac{1}{2}$
  - (b)  $\frac{3}{8}$
  - (c)  $\frac{1}{8}$
  - (d) None of these
25. A bag contains 10 red and 10 green balls. A ball is drawn from it. The probability that it will be green is
- (a)  $\frac{1}{10}$
  - (b)  $\frac{1}{3}$
  - (c)  $\frac{1}{2}$
  - (d) None of these

**Answer:****Multiple Choice Questions (MCQ)**

1.	2.	3.	4.	5.	6.	7.	8.	9	10.
(b)	(d)	(c)	(d)	(a)	(b)	(c)	(a)	(a)	(a)
11.	12.	13.	14.	15.	16.	17.	18.	19	20.
(c)	(a)	(d)	(b)	(b)	(c)	(d)	(a)	(b)	(b)
21.	22.	23.	24.	25.					
(b)	(a)	(a)	(b)	(c)					

# INDEX NUMBERS AND TIME SERIES

# 8

**This Module includes**

- 8.1 Uses of Index Numbers, Methods of Construction of Index Number**
- 8.2 Components of Time Series and Calculation of Trend by Moving Average Method**

# INDEX NUMBERS AND TIME SERIES

## **Module Learning Objectives:**

After studying this Module, the students will be able to understand –

- Concept of Index Numbers and Time Series
- Different computation methods and interpretation of results
- Practical Application of Index Number and Time Series

# Uses of Index Numbers, Methods of construction of index number

## INDEX NUMBER

Index number is a single ratio (usually in percentage) which measures the combined (i.e. averaged) change of several variables between two different times, places or situations.

The statement that the index number of wholesale prices in an economy for year 2020 was 250 compared to 2010 means that there was a net increase in the price of wholesale commodities in that economy to the extent of 150 percent.

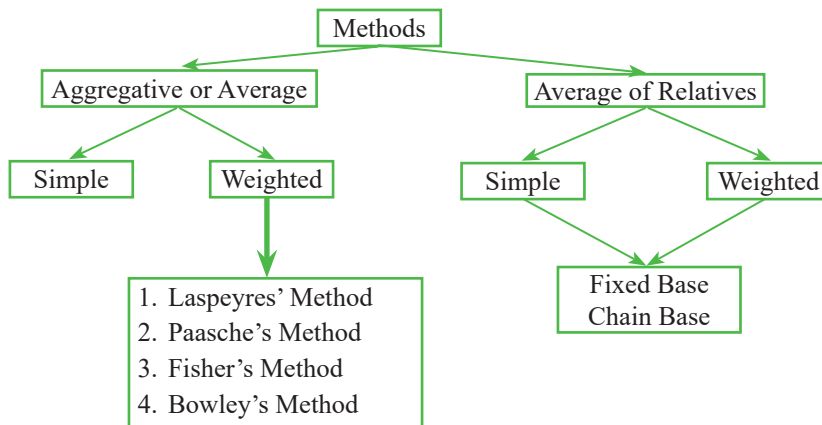
In comparing the prices of 2020 with the price of 2010, Year 2010 is taken as the *Base year*. The price at the base year is taken as 100 and the price at 2020, called *Current year*, is expressed as a percentage of the price of the base year. Thus if  $p_1$  is the price of 2020 and  $p_0$  is the price of 2010 then we can form an index number, named *Price Index* as

$$\frac{P_1}{P_0} \times 100 = \frac{\text{Current Price}}{\text{Base Price}} \times 100$$

There are various kinds of index numbers. These are broadly

- (a) Price Index
- (b) Quantity Index
- (c) Value Index
- (d) Diffusion Index
- (e) Others

### Methods of computation of Index Numbers:



**PRICE INDEX:****Aggregative Method:**

(a) Simple average or simple aggregative method

$$P_{01} = \frac{\sum P_1}{\sum P_0} \times 100$$

Where

 $P_{01}$  = Price index Number for the current year 1 w.r.t the base year 0. $P_1$  = Price for the current period $P_0$  = Price for the base period**Illustration 1.** From the following details calculate price index using simple aggregative method:

Commodities	Units	Base Year Price ( $P_0$ in ₹)	Current Year Price ( $P_1$ in ₹)	Current Year Price ( $P_2$ in ₹)
Butter	kg	20	23	36
Biscuits	kg	10	14	19.5
Eggs	Dozen	3.6	4.25	6
Vegetable oil	kg	8	10.75	18
Bread	400 gms	0.7	0.95	1.25
Total	—	42.3	52.95	80.75

$$P_{01} = \frac{\sum P_1}{\sum P_0} \times 100 = \frac{52.95}{42.3} \times 100 = 125.177$$

$$P_{02} = \frac{\sum P_2}{\sum P_0} \times 100 = \frac{80.75}{42.3} \times 100 = 190.898$$

This method though simple has the following limitations:

- The prices of various commodities may be quoted in different units. Thus the index is influenced very much by the units in which commodities are quoted and accordingly some of the commodities may get more importance because they are quoted in a particular unit.
- In this method various commodities are weighted according to the magnitudes of their prices and accordingly commodities which are highly priced exert a greater influence on the value of the index than the commodities which are low priced

To overcome above difficulties appropriate weights are assigned to various commodities to reflect the relative importance in the group. Usually the quantities consumed, sold or marketed in the base year or in current year or in some typical year are used as weights. So we do weighted aggregative method as—

(b) Weighted aggregative method:

$$P_{01} = \frac{\sum P_1 W}{\sum P_0 W} \times 100$$

Where

$P_{01}$  = Price index Number for the current year 1 w.r.t the base year 0

$P_1$  = Price for the current period

$P_0$  = Price for the base period

$W$  = Weight attached to different commodities

**Illustration 2.** From the following details calculate price index using weighted aggregative method

Commodities	Quantity Units (W)	Prices		$P_0 W$	$P_1 W$
		$P_0$	$P_1$		
A	7	0.321	0.581	2.247	4.067
B	8	0.054	0.067	0.432	0.536
C	4	0.224	0.305	0.896	1.22
Total	—	—	—	3.575	5.823

$$P_{01} = \frac{\sum P_1 W}{\sum P_0 W} \times 100 = \frac{5.823}{3.575} \times 100 = 162.88$$

(b.1) Laspeyres' Method:

$$P_{01} = \frac{\sum P_1 q_0}{\sum P_0 q_0} \times 100$$

Where

$P_{01}$  = Price index Number for the current year 1 w.r.t the base year 0

$P_1$  = Prices for the current period

$P_0$  = Prices for the base period,  $q_0$  = Quantity of base period

**Illustration 3.** From the following details calculate price index using Laspeyres' Method:

Commodity	Price (₹)		Quantity	
	Base ( $P_0$ )	Current ( $P_1$ )	Base ( $Q_0$ )	Current ( $Q_1$ )
A	2	2.5	10	12
B	1.8	3.2	16	10
C	3.5	4.8	8	8
D	2.8	4	12	10

**Solution:**

**Calculations for Laspeyres' Price Index Number**

Commodity	$P_1$	$P_0$	$Q_1$	$Q_0$	$P_1 \times Q_0$	$P_0 \times Q_0$
A	2.5	2	12	10	25	20
B	3.2	1.8	10	16	51.2	28.8
C	4.8	3.5	8	8	38.4	28
D	4	2.8	10	12	48	33.6
Total					162.6	110.4

$$\text{Laspeyres' Index} = P_{01} = \frac{\sum P_1 q_0}{\sum P_0 q_0} \times 100 = \frac{162.6}{110.4} \times 100 = 147.28$$

(b.2) Paasche's Method:

$$P_{01} = \frac{\sum P_1 q_1}{\sum P_0 q_1} \times 100$$

**Illustration 4.** From the data of the above illustration calculate price index using Paasche's Method:

**Solution:**

**Calculations for Paasche's Price Index Number**

Commodity	$P_1$	$P_0$	$Q_1$	$Q_0$	$P_1 \times Q_1$	$P_0 \times Q_1$
A	2.5	2	12	10	30	24
B	3.2	1.8	10	16	32	18
C	4.8	3.5	8	8	38.4	28
D	4	2.8	10	12	40	28
Total					140.4	98

$$\text{Paasche's Index} = P_{01} = \frac{\sum P_1 q_1}{\sum P_0 q_1} \times 100 = \frac{140.4}{98} \times 100 = 143.26$$

(b.3) Fisher's Method:

$$P_{01} = \sqrt{L \times P} = \sqrt{\frac{\sum P_1 q_0}{\sum P_0 q_0} \times \frac{\sum P_1 q_1}{\sum P_0 q_1}} \times 100$$

Where:

$L$  = Price index by Laspeyres' Method

$P$  = Price index by Paasche's Method

**Illustration 5.** From the data of the above illustration calculate price index using Fisher's Method:

**Solution:**

$$\text{Fisher's Price Index} = P_{01} = \sqrt{L \times P} = \sqrt{\frac{\sum P_1 q_0}{\sum P_0 q_0} \times \frac{\sum P_1 q_1}{\sum P_0 q_1}} \times 100 = \sqrt{1.4728 \times 1.4326} \times 100 = 145.26$$

(b.4) Bowley's Method:

$$P_{01} = \frac{L+P}{2} \times 100 = \frac{\sum P_1q_0 + \sum P_0q_1}{\sum P_0q_0 + \sum P_0q_1} \times 100$$

**Illustration 6.** From the data of the above illustration calculate price index using Bowley's Method:

**Solution:**

$$\text{Bowley's Price Index} = P_{01} = \frac{L+P}{2} \times 100 = \frac{\sum P_1q_0 + \sum P_0q_1}{\sum P_0q_0 + \sum P_0q_1} \times 100 = \frac{1.4728 + 1.4326}{2} \times 100 = 145.27$$

#### Difference between Laspeyres' and Pasche's Method:

Although both the methods follow Weighted Aggregative method but there are differences:

- In Laspeyres' method since the quantities of the base year are used as weights, the influence of price changes on quantity demanded would not get reflected in index numbers.
- But this would be reflected in Pasche's method where the quantities of the current year are used as weights.
- Laspeyres' method represents the cost of maintaining the same rate of consumption or production as in the base year but at current year's price, whereas Pasche's method represents the cost of consumption or production as a whole in the current year as compared with that in the base year. i.e. effects of changes in prices on the quantity consumed in the current period do not get reflected in Laspeyres' method but get reflected in Pasche's method.
- Laspeyres' method is basically upward biased whereas Pasche's method is downward biased

To overcome these difficulties Fisher's and Bowley's methods are applied to take into account the influence of current as well as the base year.

Fisher's method is known as Ideal due to the following reasons:

- It is free from bias, since the upward bias of Laspeyres' method is balanced to a great extent by the downward bias of Pasche's method
- This method takes into account the influence of the current as well as the base year
- It is based on the geometric mean, theoretically which is considered to be the best average for constructing index numbers

#### Relatives Method:

Under this method the price of each commodity in the current year is expressed as a percentage of the price of the base year. This is known as "Price Relative" and is given by the formula

$$\text{Price Relative of a commodity} = \frac{\text{Price of the commodity in the current year}}{\text{Price of the commodity in the base year}} \times 100$$

(a) Simple average of Relatives index:

- Using AM:  $P_{01} = \frac{1}{n} = \sum \left( \frac{P_1}{P_0} \times 100 \right)$
- Using GM:  $P_{01} = \text{Antilog} \left[ \sum \log(P) / n \right]$

Where

$P_1$  = Price of a commodity in the current year

$P_0$  = Price of a commodity in the Base year

$P$  = Price Relative =  $\frac{P_1}{P_0} \times 100$

$n$  = number of commodities

(b) Weighted A.M of Relatives index:

$$P_{01} = \frac{\sum Pw}{\sum w} = \frac{\sum \left[ \left( \frac{P_1}{P_0} \times 100 \right) w \right]}{\sum w}$$

Similarly we can construct different quantity index as follows:

### QUANTITY INDEX:

(a) Simple aggregative quantity index:

$$Q_{01} = \frac{\sum q_1}{\sum q_0} \times 100$$

(b) Laspeyres' Quantity Index:

$$Q_{01} = \frac{\sum q_1 P_0}{\sum q_0 P_0} \times 100$$

(c) Paasche's Quantity Index:

$$Q_{01} = \frac{\sum q_1 P_1}{\sum q_0 P_1} \times 100$$

(d) Fisher's Quantity Index:

$$Q_{01} = \sqrt{L \times P} = \sqrt{\frac{\sum q_1 P_0}{\sum q_0 P_0} \times \frac{\sum q_1 P_1}{\sum q_0 P_1}} \times 100$$

(e) Simple average of Relatives index:

Using AM:  $Q_{01} = \frac{\sum Q}{n} = \frac{1}{n} = \sum \left( \frac{q_1}{q_0} \times 100 \right)$

Using GM:  $Q_{01} = \text{Antilog} \left[ \sum \log \frac{Q}{n} \right]$

(f) Weighted A.M of Relatives index:

$$Q_{10} = \frac{\sum Q_w}{w} = \frac{\sum \left[ \left( \frac{q_1}{q_0} \times 100 \right) w \right]}{\sum w}$$

Where

$q_1$  = Quantity of a commodity in current year

$q_0$  = Quantity of a commodity in Base year

$Q$  = Quantity Relative =  $\frac{q_1}{q_0} \times 100$

$n$  = number of commodities

### General Index from group indices:

The items (or commodities) included in constructing an index Number are divided into some groups – similar or related items falling in the same group. Index number for each group (called Group index) is then determined. The General Index is the Arithmetic Mean of the group indices and is given as

General Index =  $\frac{\sum wI}{\sum w}$ , where I = Group Index and w = group weight

**Illustration 7.** In 2023, the average price of a commodity was 20% more than in 2022, but 20% less than in 2021. Find price relatives by using 2022 as base.

**Solution:** Suppose the price in 2023 is ₹ 100.

$$\text{Price in 2022} = \frac{100}{120} \times 100 = 83.3$$

$$\text{Price in 2021} = \frac{100}{80} \times 100 = 125.$$

Price relatives as 2022 as base

$$\text{For 2021} = \frac{125}{83.3} \times 100 = 150$$

$$\text{For 2022} = 100$$

$$\text{For 2023} = \frac{100}{83.3} \times 100 = 120.$$

**Illustration 8.** In 2022, the average price of a commodity was 20% more than in 2021 but 50% more than in 2023. Find price relatives by using 2021 & 2022 as base year.

**Solution:** Suppose price in 2022 is ₹ 100.

$$\text{Price in 2021} = \frac{100}{100} \times 100 = 83.3$$

$$\text{Price in 2023} = \frac{100}{150} \times 100 = 66.7$$

$$\text{Average price of 2021 \& 2022} = \frac{83.3 + 100}{2} = 91.65$$

Price relatives by 91.65 as base

$$\text{For 2021} = \frac{83.3}{91.65} \times 100 = 90.88$$

$$\text{For 2022} = \frac{100}{91.65} \times 100 = 109.11$$

$$\text{For 2023} = \frac{66.7}{91.65} \times 100 = 72.77$$

**Illustration 9.** Net monthly income of an employee was ₹ 8,000 P.M in 2003. The consumer price index number was 80 in 2003. It became 300 in 2023. Calculate the additional D.A to be paid to the employee if he has to be compensated.

$$\text{Solution:} \quad = \frac{\text{Price in 2023}}{\text{Price in 2003}} = \frac{\text{Price index in 2023}}{\text{Price index in 2003}}$$

$$\text{Or, Price in 2023} = \frac{300}{80} \times 8000 = ₹ 30,000.$$

$$\text{Additional D.A} = ₹ 30,000 - ₹ 8,000 = ₹ 22,000.$$

### EXERCISE:

**Illustration 10.** From the following compute Laspeyres' & Paasche's Price index.

Item	Base Year		Current year	
	Unit Price	Quantity	Unit Price	Quantity
A (dozen)	3.6	20	5.2	15
B (litre)	1.0	30	2.4	25
C (kg)	8.00	50	12.00	40
D (kg)	2.5	100	5.00	60

**Solution:**

Laspeyres' Price Index:

#### Calculations for Laspeyres' Price Index

Commodity	$P_1$	$P_0$	$q_0$	$P_1 q_0$	$P_0 q_0$
A	5.2	3.6	20	104	72
B	2.4	1	30	72	30
C	12	8	50	600	400
D	5	2.5	100	500	250
Total	—	—	—	1276	752

$$P_{01} = \frac{\sum P_1 q_0}{\sum P_0 q_0} \times 100 = \frac{1276}{752} \times 100 = 169.68$$

Paasche's Method:

#### Calculations for Paasche's Price Index

Commodity	$P_1$	$P_0$	$q_1$	$P_1 q_1$	$P_0 q_1$
A	5.2	3.6	15	78	54
B	2.4	1	25	60	25
C	12	8	40	480	320
D	5	2.5	60	300	150
Total	—	—	—	918	549

$$P_{01} = \frac{\sum P_1 q_1}{\sum P_0 q_1} \times 100 = \frac{918}{549} \times 100 = 167.21$$

**Illustration 11.** Refer Q1. Fisher's price index is

- (a) 186.23
- (b) 168.44
- (c) 203.15
- (d) 114.98

**Solution: (b)**

$$\text{Fisher's Price Index} = P_{01} = \sqrt{L \times P} = \sqrt{\frac{\sum P_1 q_0}{\sum P_0 q_0} \times \frac{\sum P_1 q_1}{\sum P_0 q_1}} \times 100 = \sqrt{\frac{1276}{752} \times \frac{918}{549}} \times 100 = 168.44$$

**Illustration 12.** Refer Q1. Bowley's price index is

- (a) 186.23
- (b) 203.15
- (c) 168.44
- (d) 114.98

**Solution: (c)**

$$\text{Bowley's Price Index} = P_{01} = \frac{L + P}{2} \times 100 = \frac{\frac{\sum P_1 q_0}{\sum P_0 q_0} + \frac{\sum P_1 q_1}{\sum P_0 q_1}}{2} \times 100 = \frac{\frac{1276}{752} + \frac{918}{549}}{2} \times 100 = 168.44$$

**Illustration 13.** Calculate Laspeyres' and the Paasche's Quantity index from the following data.

Item	Base Year		Current Year	
	Unit Price	Quantity	Unit Price	Quantity
A	4	10	6.8	11
B	1	8	7.4	10
C	4.5	5	7.5	8

**Solution:**

#### Calculations for Laspeyres' Quantity Index

Commodity	$P_0$	$q_1$	$q_0$	$q_1 P_0$	$q_0 P_0$
A	4	11	10	44	40
B	1	10	8	10	8
C	4.5	8	5	36	22.5
Total	—	—	—	90	70.5

$$\text{Laspeyres' Quantity Index} = Q_{01} = \frac{\sum q_1 P_0}{\sum q_0 P_0} \times 100 = \frac{90}{70.5} \times 100 = 127.66$$

#### Calculations for Paasche's Quantity Index

Commodity	$P_1$	$q_1$	$q_0$	$q_1 P_1$	$q_0 P_1$
A	6.8	11	10	74.8	68
B	2.4	10	8	24	19.2
C	7.5	8	5	60	37.5
Total	—	—	—	158.8	124.7

$$\text{Paasche's Quantity Index} = Q_{01} = \frac{\sum q_1 P_1}{\sum q_0 P_1} \times 100 = \frac{158.8}{124.7} \times 100 = 127.34$$

**Illustration 14.** S limited sells three types of product A, B and C. Company's records show the prices and quantities sold of each type are as follows:

Item	Base Year		Current year	
	Unit Price	Quantity	Unit Price	Quantity
A	30	22	40	30
B	50	31	60	40
C	120	8	99	12

- Calculate an un-weighted price index for the current year
- Calculate a weighted Lapeyres' aggregate price index for the current year

**Solution:**

(a) Un-weighted price index:

$$P_{01} = \frac{\sum P_1}{\sum P_0} \times 100 = \frac{199}{200} \times 100 = 99.5$$

#### Calculations for Unweighted Price Index & Laspeyres' Aggregate Price Index

Commodity	$P_1$	$P_0$	$q_1$	$q_0$	$P_1 q_0$	$P_0 q_0$
A	40	30	30	22	880	660
B	60	50	40	31	1860	1550
C	99	120	12	8	792	960
Total	199	200	—	—	3532	3170

$$(b) \quad \text{Lapeyres' aggregate price index} = P_{01} = \frac{\sum P_1 q_0}{\sum P_0 q_0} \times 100 = \frac{3532}{3170} \times 100 = 111.42$$

**Illustration 15.** On the basis of the following information calculate the Fisher's Ideal Index Number:

Item	Base Year		Current year	
	Unit Price	Quantity	Unit Price	Quantity
A	2	40	6	50
B	4	50	8	40
C	6	20	9	30
D	8	10	6	20
E	10	10	5	20

**Solution:**

$$\text{Fisher's Ideal Index No.} = P_{01} = \sqrt{L \times P} = \sqrt{\frac{\sum P_1 q_0}{\sum P_0 q_0} \times \frac{\sum P_1 q_1}{\sum P_0 q_1}} \times 100$$

**Calculations for Fisher's Ideal Index Number**

Commodity	$P_1$	$P_0$	$q_1$	$q_0$	$P_1 q_0$	$P_0 q_0$	$P_1 q_1$	$P_0 q_1$
A	6	2	50	40	240	80	300	100
B	8	4	40	50	400	200	320	160
C	9	6	30	20	180	120	270	180
D	6	8	20	10	60	80	120	160
E	5	10	20	10	50	100	100	200
Total	—	—	—	—	930	580	1110	800

So

$$P_{01} = \sqrt{L \times P} = \sqrt{\frac{\sum P_1 q_0}{\sum P_0 q_0} \times \frac{\sum P_1 q_1}{\sum P_0 q_1}} \times 100 = \sqrt{\frac{930}{580} \times \frac{1110}{800}} \times 100 = 149.16$$

**Illustration 16.** Compute a price index from the following data by (a) simple aggregative method and (b) average of price relatives method (using both arithmetic mean and geometric mean):

Commodity	A	B	C	D	E	F
Price in Base Year	20	30	10	25	40	50
Price in Current Year	25	30	15	35	45	55

**Solution:**

**Calculations for Simple Aggregative Index & Simple Average of Relatives Index**

Commodity	$P_0$	$P_1$	Price Relative = $\frac{P_1}{P_0} \times 100$	Log P
A	20	25	125	2.09691
B	30	30	100	2
C	10	15	150	2.176091

D	25	35	140	2.146128
E	40	45	112.5	2.051153
F	50	55	110	2.041393
Total	175	205	737.5	12.511675

(a) Simple Aggregative Method:

$$\text{Index Price} = P_{01} = \frac{\sum P_1}{\sum P_0} \times 100 = \frac{205}{175} \times 100 = 117.14$$

(b) Simple Average of Relatives Index:

Here  $n$  = No. of commodities = 6

► Using AM:  $P_{01} = \frac{1}{n} \sum \left( \frac{P_1}{P_0} \times 100 \right) = \frac{1}{6} \times 737.5 = 122.92$

► Using GM:  $P_{01} = \text{Antilog} \left[ \frac{(\sum \log P)}{n} \right] = \text{Antilog} \left[ \frac{12.511675}{6} \right] = 121.7$

**Illustration 17.** From the following data prepare Laspeyres', Paasche's and Fisher's quantity index numbers for the year 2 taking year 1 as the base

Item	Year 1		Year 2	
	Unit Price	Quantity	Unit Price	Quantity
I	5	10	4	12
II	8	6	7	7
III	6	3	5	4

**Solution:**

Commodity	$P_1$	$P_0$	$q_1$	$q_0$	$q_1 P_0$	$q_0 P_0$	$q_1 P_1$	$q_0 P_1$
I	4	5	12	10	60	50	48	40
II	7	8	7	6	56	48	49	42
III	5	6	4	3	24	18	20	15
Total	—	—	—	—	140	116	117	97

For the sake of simplicity computations are made assuming Year 1 as Year 0 and Year 2 as Year 1

Laspeyres' Quantity Index:

$$Q_{01} = \frac{\sum q_1 P_0}{\sum q_0 P_0} \times 100 = \frac{140}{116} \times 100 = 120.69$$

Paasche's Quantity Index:

$$Q_{01} = \frac{\sum q_1 P_1}{\sum q_0 P_1} \times 100 = \frac{117}{97} \times 100 = 120.62$$

Fisher's Quantity Index:

$$Q_{01} = \sqrt{L \times P} = \sqrt{\frac{\sum q_1 P_0}{\sum q_0 P_0} \times \frac{\sum q_1 P_1}{\sum q_0 P_1}} \times 100 = \sqrt{1.2069 \times 1.2068} \times 100 = 120.65$$

**Illustration 18.** Compute the index number of business activity from the following data:

Industrial Activity	Index	Weight
Industrial Production	250	36
Mineral Production	135	7
Internal Trade	200	24
Financial Activity	135	20
Exports & Imports	325	7
Shipping Activity	300	6

**Solution:**

Detail computations are:

Industrial Activity	Index (I)	Weight (W)	IW
Industrial Production	250	36	9000
Mineral Production	135	7	945
Internal Trade	200	24	4800
Financial Activity	135	20	2700
Exports & Imports	325	7	2275
Shipping Activity	300	6	1800
<b>Total</b>	—	<b>100</b>	<b>21520</b>

So required index number of business activities is given by *arithmetic mean* method as

$$= \frac{\sum IW}{\sum W} = \frac{21520}{100} = 215.20$$

**Problems in the construction of Index Numbers:**

- (a) Determination of the exact purpose for which proposed index is constructed
- (b) Selection of items which are to be included in the group of items/commodities
- (c) Collection of data regarding price and quantities and selection of sources from which such data are to be collected
- (d) Choice of the base period
- (e) Determination of the weights to be employed
- (f) Selection between AM and GM

# Components of Time Series and Calculation of Trend by Moving Average Method

## 8.2

A time series is a collection of data recorded over a period of time--- like data collected daily, data collected quarterly, data collected monthly or data collected yearly.

e.g. daily stock price movements in national stock exchange (given as follows) form a time series:

Symbol	Series	Date	Prev. Close	Open Price	High Price	Low Price
ITC	EQ	3-Feb-22	232.15	232.15	235.2	231.8
ITC	EQ	4-Feb-22	234.45	237.25	238.5	231.55
ITC	EQ	7-Feb-22	234.3	234	234.75	228.5
ITC	EQ	8-Feb-22	230.2	230.3	232.3	229.05
ITC	EQ	9-Feb-22	231.25	232.3	233	228.75
ITC	EQ	10-Feb-22	230.15	228.7	233.4	227.75
ITC	EQ	11-Feb-22	232.25	232	233.3	229.8
ITC	EQ	14-Feb-22	232.45	226	226	218.5
ITC	EQ	15-Feb-22	219.45	221	223.5	218.55
ITC	EQ	16-Feb-22	223	223.8	224.7	221.85
ITC	EQ	17-Feb-22	222.75	223.05	224.35	220.85
ITC	EQ	18-Feb-22	222.65	222.65	223.45	221.5
ITC	EQ	21-Feb-22	221.9	220.3	221.3	217.15
ITC	EQ	22-Feb-22	218	215	216.9	213
ITC	EQ	23-Feb-22	214.85	216.8	217.5	215.05

Analysis of time series means an analysis of history and is used by management to make current decisions and plans based on long term forecasting.

### Components of a Time series:

There are four components in time series:

- ▶ The trend (T)
- ▶ The cyclical fluctuation (C)
- ▶ The seasonal variation (S)
- ▶ The irregular movement (I)

The four components of a time series referred above can be considered to be interacting in *additive or multiplicative* fashion to produce the observed values in a time series (here daily closing prices of ITC shares)

- ▶ Under additive model Observed Values ( $Y$ ) = Trend ( $T$ ) + Cyclical Fluctuation ( $C$ ) + Seasonal Variation ( $S$ ) + Irregular Movement ( $I$ )
- ▶ Here  $S$ ,  $C$  and  $I$  are absolute quantitative deviations about trend
- ▶ Under multiplicative model Observed Values ( $Y$ ) = Trend ( $T$ ) × Cyclical Fluctuation ( $C$ ) × Seasonal Variation ( $S$ ) × Irregular Movement ( $I$ )
- ▶ Here  $S$ ,  $C$  and  $I$  are rates, percentages or index numbers
- ▶ Under additive model it is assumed that all the four components of the time series operate independently of one another whereas under multiplicative model it is assumed that the four components although due to different causes are not necessarily independent and they can affect one another.
- ▶ Additive model is not appropriate for extending its application to future events but the relative values of factors under multiplicative model are appropriate for forecasting future behaviour in a time series.

Most of the time series relating to economic and business phenomena conform to the multiplicative model and in practice additive model occurs rarely. So in this module we discuss time series with multiplicative model.

By now we understood what is called time series, what are its components etc. Now we will learn *decomposition* of a time series. Decomposition of a time series refers to the segregation of various components. In the process of decomposition often the trend has to be found out first. This is because most of the other components especially short term Seasonal, Cyclical and Irregular can be separated only with reference to Trend

$$\frac{\text{Observed Value (TSCI)}}{\text{Trend (T)}} = SCI$$

If we plot the closing price of ITC shares with price on the Y-axis and date on X-axis, we get a plot of time series as follows:



**Chart-I**

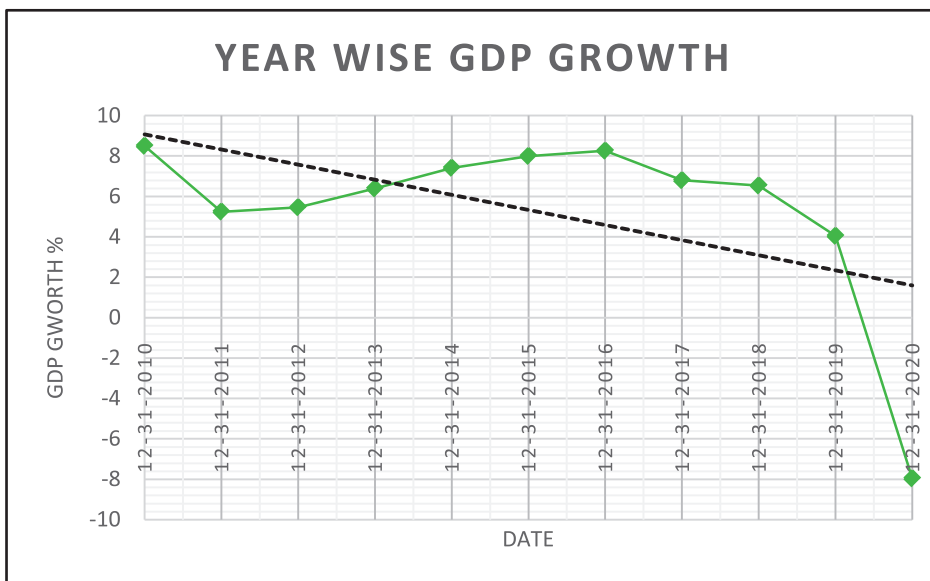
The green part of the chart-I is the time series plot. As seen there are random fluctuations date wise. Sometimes price goes above the black line and sometimes goes below that line. The black line is called *Trend or Secular trend* and it is obtained by smoothening the fluctuations in time series.

### Secular Trend: The smoothed long term direction of a time series

Although in Chart-I only a small time period is considered actually long term means more than 1 year.

India's year wise GDP growth (%) is given in the following table and corresponding time series graph is shown in Chart II with the trend line

Date	GDP Growth (%)
12/31/2010	8.4976
12/31/2011	5.2413
12/31/2012	5.4564
12/31/2013	6.3861
12/31/2014	7.4102
12/31/2015	7.9963
12/31/2016	8.2563
12/31/2017	6.7954
12/31/2018	6.533
12/31/2019	4.0416
12/31/2020	-7.9646



**Chart II**

Again we observed that there are fluctuations in time series and smoothing out all the fluctuations we get a secular trend –the black line

**Moving Average:**

A *Moving Average* is used in smoothing a time series to see its trend. Usually Moving average will incorporate cyclical movements also and therefore through moving average we basically do the following:

$$\frac{\text{Observed Value (TSCI)}}{\text{Moving Average (Trend T \& Cyclical Fluctuation C)}} = SI$$

The moving average merely smoothen the fluctuations in the data. This is accomplished by moving the arithmetic mean values through the time series.

This is illustrated with the following example.

Year	Sales (Rs. in Million)	Year	Sales (Rs. in Million)
1988	1	2001	4
1989	2	2002	3
1990	3	2003	4
1991	4	2004	5
1992	5	2005	6
1993	4	2006	7
1994	3	2007	6
1995	2	2008	5
1996	3	2009	4
1997	4	2010	5
1998	5	2011	6
1999	6	2012	7
2000	5	2013	8

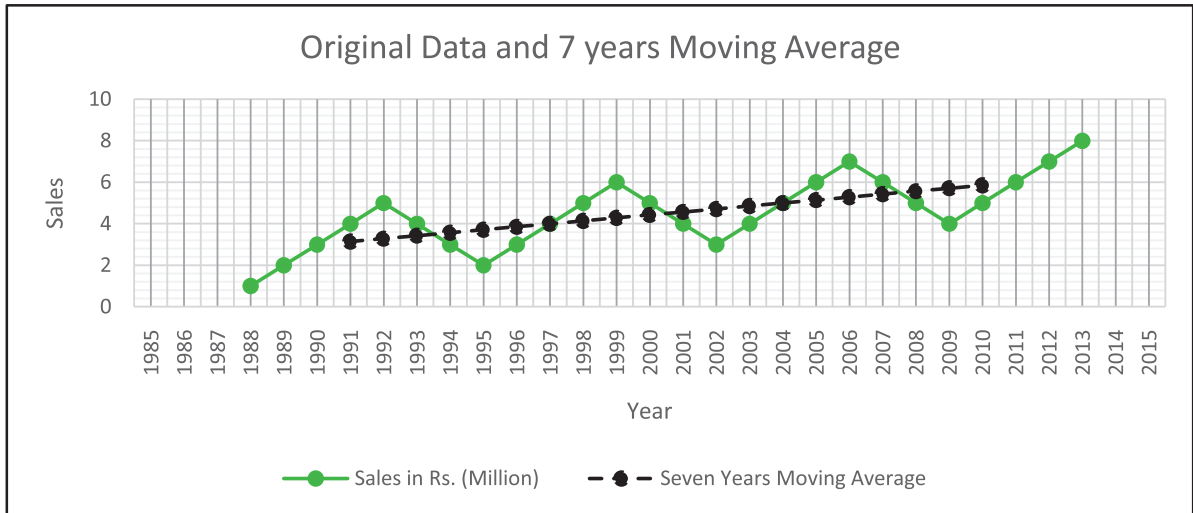
Our objective is to smooth out these fluctuations. The method we use is “*n-years Moving Average*” and the resultant smoothed out figure we usually call Secular trend which indicates the long term direction of the variable (here sales). Here  $n$  we have taken as 7. But value of  $n$  depends on the data collected. If the data are quarterly then 4 quarters moving average is typical as there are 4 quarters in a year. If the data are daily then seven days moving average is appropriate because there are 7 days in a week. We can also fix through use of trial and error the value of  $n$  that best levels out the chance fluctuations.

- The first step in finding out 7 years moving average is to find 7 years moving total
  - ▶ The total sales for the first 7 years is  $1+2+3+4+5+4+3 = 22$
  - ▶ The average of this total =  $22/7 = 3.143$
  - ▶ The 7 years total and its average are positioned against the middle year of the group i.e.1991 under the columns seven years moving total and seven years moving average respectively
- Then we form second group of 7 years starting from 1989, i.e. omitting 1988 & including 1995
  - ▶ The total sales for the second 7 years is  $2+3+4+5+4+3+2 = 23$
  - ▶ The average of this total =  $23/7 = 3.286$
  - ▶ This 7 years total and its average are positioned against the middle year of the group i.e.1992 under the columns seven years moving total and seven years moving average respectively.

- In this way we repeat the process till we form our last 7 years group which starts from 2007 and ends on 2013
  - ▶ The total sales for the last 7 years is  $6+5+4+5+6+7+8 = 41$
  - ▶ The average of this total =  $41/7 = 5.857$
  - ▶ This 7 years total and its average are positioned against the middle year of the group i.e.2010 under column seven years moving total and seven years moving average

The detail computations are in the following table. Given data of time series and 7 years moving average data are plotted in a graph shown as Chart III.

Year	Sales (₹ in Million)	Seven years Moving Total	Seven years Moving Average
1988	1	–	–
1989	2	–	–
1990	3	–	–
1991	4	22	3.143
1992	5	23	3.286
1993	4	24	3.429
1994	3	25	3.571
1995	2	26	3.714
1996	3	27	3.857
1997	4	28	4.000
1998	5	29	4.143
1999	6	30	4.286
2000	5	31	4.429
2001	4	32	4.571
2002	3	33	4.714
2003	4	34	4.857
2004	5	35	5.000
2005	6	36	5.143
2006	7	37	5.286
2007	6	38	5.429
2008	5	39	5.571
2009	4	40	5.714
2010	5	41	5.857
2011	6	–	–
2012	7	–	–
2013	8	–	–



**Chart III**

The black line indicates the 7 years moving average data as per computation table and it is observed that fluctuations in time series are smoothed out.

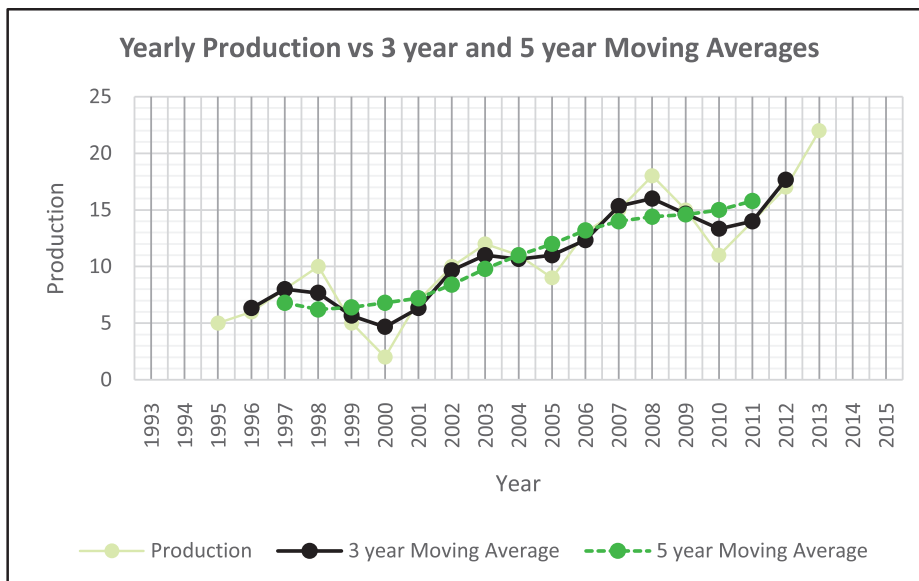
As already mentioned in “ $n$  years moving average” the value of  $n$  depends on the data collected or we can use trial and error. Let us take the following example:

Yearly production data of a business unit are as follows:

Year	Production	Year	Production
1995	5	2005	9
1996	6	2006	13
1997	8	2007	15
1998	10	2008	18
1999	5	2009	15
2000	2	2010	11
2001	7	2011	14
2002	10	2012	17
2003	12	2013	22
2004	11		

If we smooth out the fluctuation with “ $n$  years moving average” where  $n = 3$  or  $5$  i.e. if we do 3 or 5 years moving average then the detail computation s and corresponding plot will be as follows:

Year	Production	3-year Moving Total	3-year Moving Average	5-year Moving Total	5-year Moving Average
1995	5	–	–	–	–
1996	6	19	6.333	–	–
1997	8	24	8.000	34	6.8
1998	10	23	7.667	31	6.2
1999	5	17	5.667	32	6.4
2000	2	14	4.667	34	6.8
2001	7	19	6.333	36	7.2
2002	10	29	9.667	42	8.4
2003	12	33	11.000	49	9.8
2004	11	32	10.667	55	11
2005	9	33	11.000	60	12
2006	13	37	12.333	66	13.2
2007	15	46	15.333	70	14
2008	18	48	16.000	72	14.4
2009	15	44	14.667	73	14.6
2010	11	40	13.333	75	15
2011	14	42	14.000	79	15.8
2012	17	53	17.667	–	–
2013	22	–	–	–	–



**Chart IV**

Since 5 years moving average reduces the fluctuation in a much better way than that done by 3 years moving average so with the given production data we should take 5 years moving average.

In chart III perfectly linear trend is observed but in chart V moving average does not result precisely in a straight line. This is because (i) periods of oscillation do not have equal length (ii) oscillation do not have identical amplitudes. Sales, production and other economic and business time series data usually generate not precise linear trend through moving average but trend can always give long term direction of movement like upward, downward etc.

Four year, six-year and other even-numbered-year moving averages present one minor problem regarding positioning the moving totals and moving averages. Let us take the following example:

Sales figures of a business unit are as follows:

2005	2006	2007	2008	2009	2010	2011	2012	2013
8	11	9	14	9	10	10	8	12

Let us compute 4 years moving average. Details are in the following table:

Year	Sales	4 year Moving Total	4 year Moving Average	4 year Centered Moving Average
2005	8	–	–	–
2006	11	–	–	–
		42	10.5	
2007	9			10.625
		43	10.75	
2008	14			10.625
		42	10.5	
2009	9			10.625
		43	10.75	
2010	10			10
		37	9.25	–
2011	10			9.625
		40	10	
2012	8	–	–	–
2013	12	–	–	–

- 1<sup>st</sup> Moving total comprises of the data for 2005, 2006, 2007 and 2008 i.e. 8, 11, 9 and 14 respectively
- 1<sup>st</sup> Moving total =  $(8+11+9+14) = 42$
- Since there is no particular year which can be taken as center time period, 42 is positioned between two central time periods i.e. 2006 and 2007

- Same procedure is applied for 2<sup>nd</sup>, 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup> and 6<sup>th</sup> moving total i.e. with 43 (11+9+14+9), 42 (9+14+9+10), 43 (14+9+10+10), 37 (9+10+10+8), 40 (10+10+8+12)
- 4 years moving average i.e. 10.5 (42/4), 10.75 (43/4), 10.5 (42), 10.75 (43/4), 9.25 (37/4) & 10 (40/4) are accordingly placed between two time periods
- Now average of 1<sup>st</sup> four years i.e.10.5 and average of 2<sup>nd</sup> four years i.e.10.75 are averaged and this average 10.625 [(10.5+10.75)/2] is centered on 2007.
- Similarly we will get centered moving average against year 2008, 2009, 2010, and 2011.

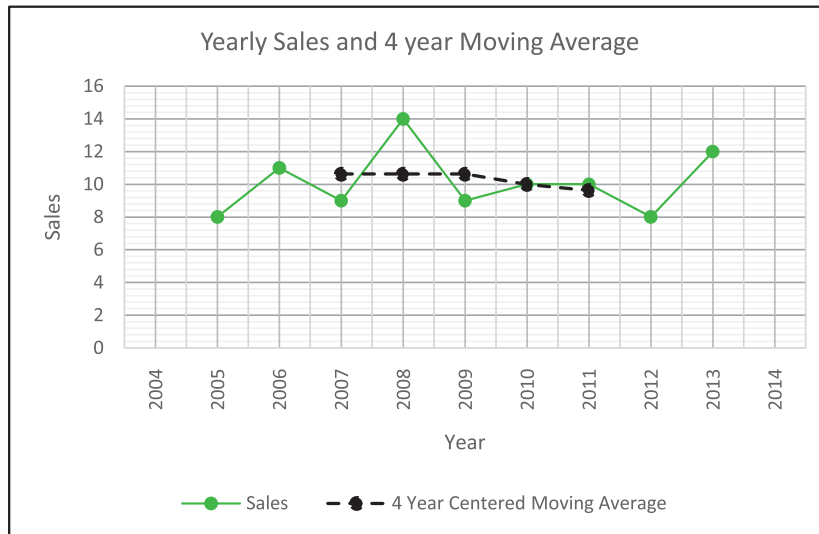


Chart V

**Weighted Moving Average:** A moving average uses the same weight for each observation. But weighted moving average involves selecting a different weight for each data value and then computing a weighted average of the most recent “ $n$ ” values as the smoothed value. Most recent observation receives the most weight and the weight decreases for older data values. Sum of weights must be equal to 1.

Let us take one example. Yearly sales of a business unit are given below in ₹ '000:

Year	Sales	Year	Sales
1993	5761	2003	12181
1994	6148	2004	12557
1995	6783	2005	12700
1996	7445	2006	19300
1997	7405	2007	22100
1998	11450	2008	22720
1999	11224	2009	21136
2000	11703	2010	22785
2001	11890	2011	23377
2002	12380	2012	23300

We are going to find 3 years weighted moving average with weights 0.2, 0.3 and 0.5 respectively.

1<sup>st</sup> one of the 3 years weighted moving total  $0.2 \times 5761 + 0.3 \times 6148 + 0.5 \times 6388.1$  Most recent values are given more weight and oldest value is given the least weight. Sum of the weights must be equal to 1. For the given case the detail computations are:

Year	Sales (₹ '000)	3 years weighted Moving Total	3 years weighted Moving Average **
2003	5761	–	–
2004	6148	6388.1	6388.1
2005	6783	6987	6987
2006	7445	7292.6	7292.6
2007	7405	9435.5	9435.5
2008	11450	10528	10528
2009	11224	11508.7	11508.7
2010	11703	11700.7	11700.7
2011	11890	12097.6	12097.6
2012	12380	12182.5	12182.5
2013	12181	12408.8	12408.8
2014	12557	12553.3	12553.3
2015	12700	15971.4	15971.4
2016	19300	19380	19380
2017	22100	21850	21850
2018	22720	21804	21804
2019	21136	22277.3	22277.3
2020	22785	22751.2	22751.2
2021	23377	23220.1	23220.1
2022	23300	–	–

\*\* since sum of weights is equal to 1 we will always get weighted moving total and weighted moving average same value

1<sup>st</sup> weighted moving total  $0.2 \times 5761 + 0.3 \times 6148 + 0.5 \times 6783 = 6388.1$

So weighted moving average  $= \frac{6388.1}{0.2+0.3+0.5} = \frac{6388.1}{1} = 6388.1$

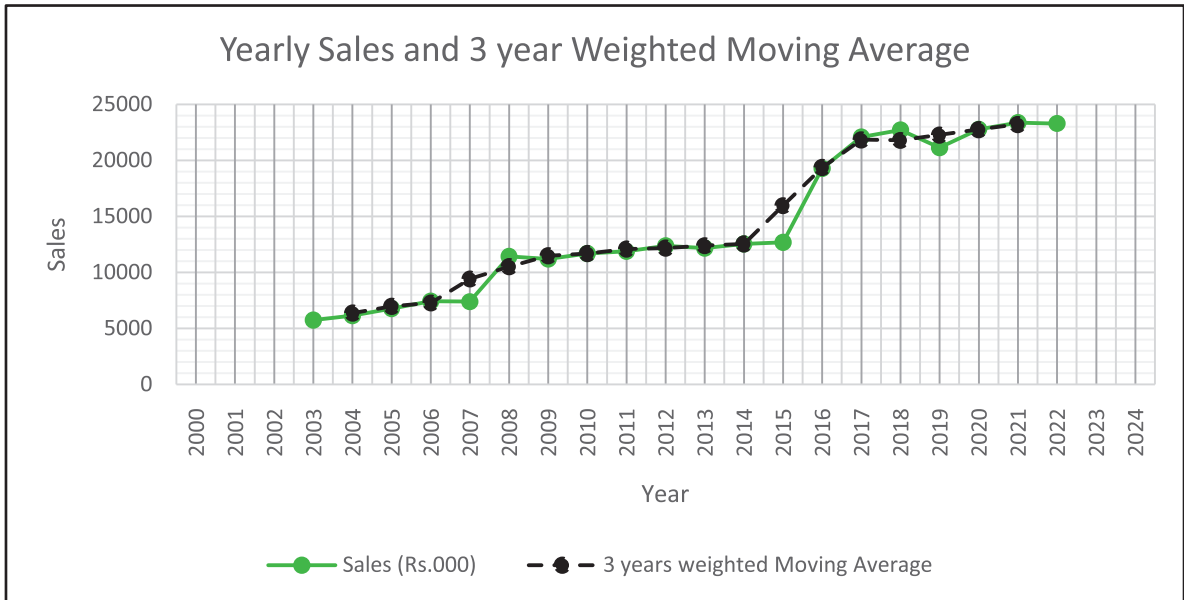


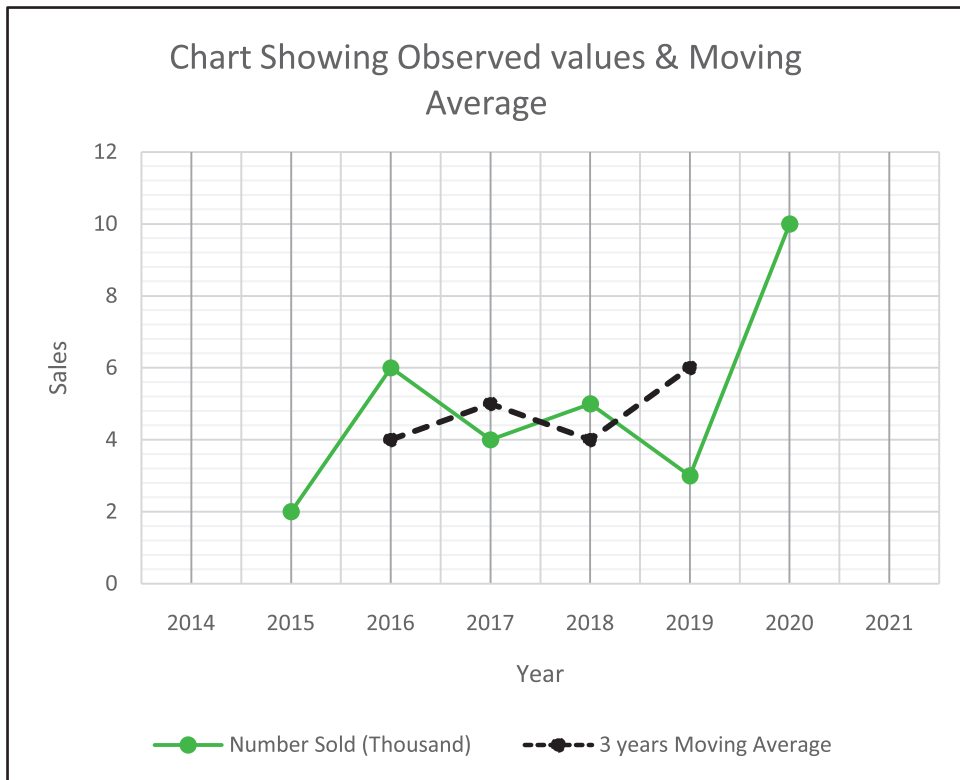
Chart VI

**Illustrations**

**Illustration 19.** Determine a three year moving average for the sales of a Maruti Udyog limited.

**Solution:**

Year	Number sold (Thousand)	3 years Moving Total	3 years Moving Average
2015	2		
2016	6	12	4
2017	4	15	5
2018	5	12	4
2019	3	18	6
2020	10		



**Illustration 20.** Calculate a four-quarter weighted moving average for the number of shares outstanding for a company for the nine quarters of data. The data are reported in thousands. Apply weights of 0.1, 0.2, 0.3 and 0.4 respectively for Q1, Q2, Q3 & Q4.

Year	Quarter	No of shares Outstanding
2015	Q1	28766
	Q2	30057
	Q3	31336
	Q4	33240
2016	Q1	34610
	Q1	35102
	Q1	35308
	Q1	35203
2017	Q1	34386

**Solution:**

1<sup>st</sup> four quarter weighted moving average is 31584.8

It is found by  $(28766 \times 0.1 + 30057 \times 0.2 + 31336 \times 0.3 + 33240 \times 0.4) / (0.1 + 0.2 + 0.3 + 0.4)$

Similarly for other values. Details are as follows:

Year	Quarter	No of shares Outstanding	4 Quarter weighted Moving Average	4 Quarter weighted Moving Average
2015	Q1	28766	–	–
	Q2	30057	–	–
			31584.8	
	Q3	31336		32336.85
			33088.9	
	Q4	33240		33647.15
			34205.4	
2016	Q1	34610		34552.6
			34899.8	
	Q2	35102		35027.4
			35155	
	Q3	35308		35021.05
			34887.1	
	Q4	35203	–	–
2017	Q1	34386	–	–

**Illustration 21.** Using three year moving average determine the trend and short term fluctuations. Plot the original and trend values.

Year	Productions Units (Thousand)
2010	21
2011	22
2012	23
2013	25
2014	24
2015	22
2016	25
2017	26
2018	27
2019	26

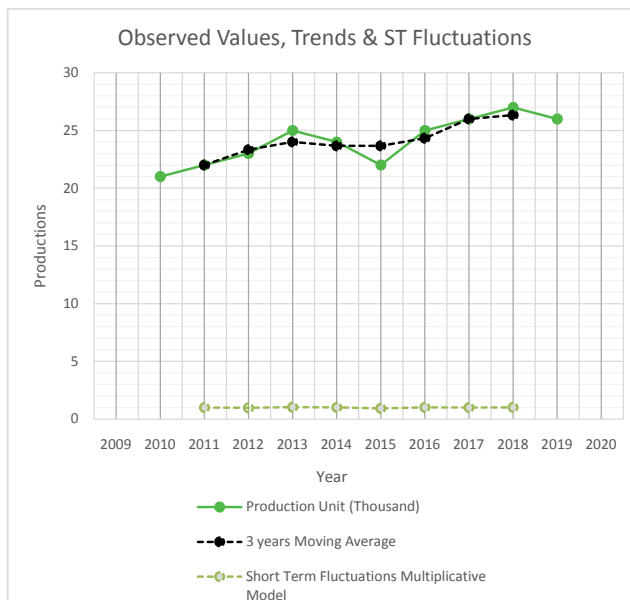
**Solution:**

In this problem 3 years moving averages are calculated as usual. And these three years moving averages are the Trend values.

For finding out short term fluctuations we usually use Multiplicative model. As discussed earlier Observed values contain TSCI (i.e. Trend, Seasonal, Cyclical and Irregular components). Moving average values contain Trend (T)

So short term fluctuations under multiplicative model =  $\frac{\text{Observed Values}}{\text{Moving average}}$  & short term fluctuations under additive model =  $\text{Observed Values} - \text{Moving average}$ . Detail computations are:

Year	Productions Unit (Thousand)	3 years Moving Total	3 years Moving Average	Short Term Fluctuations Multiplicative Model	Short Term Fluctuations Additive Model
2010	21	—	—	—	—
2011	22	66	22.000	1	0.000
2012	23	70	23.333	0.986	-0.333
2013	25	72	24.000	1.042	1.000
2014	24	71	23.667	1.014	0.333
2015	22	71	23.667	0.930	-1.667
2016	25	73	24.333	1.027	0.667
2017	26	78	26.000	1.000	0.000
2018	27	79	26.333	1.025	0.667
2019	26	—	—	—	—



**Illustration 22.** The following data give daily sales of a shop observing a five day week, over four successive weeks. Determine the period of the moving average and calculate the moving average accordingly:

Day	1	2	3	4	5	6	7	8	9	10
Sales	26	29	35	47	51	26	32	37	46	53
Day	11	12	13	14	15	16	17	18	19	20
Sales	28	30	36	46	54	28	31	36	46	54

**Solution:**

Determination of the period of moving average: We discussed earlier in finding out “*n-years Moving Average*” value of  $n$  depends on the data collected. If the data are quarterly then 4 quarters moving average is typical as there are 4 quarters in a year. If the data are daily then seven days moving average is appropriate because there are 7 days in a week. We can also fix by trial and error the value of  $n$  that best levels out the chance fluctuations.

In the given case data is collected daily in a week and week is composed of 5 days. So 5 days moving average is appropriate and the detail computations are as follows:

Days	Sales (Thousand ₹)	5 day Moving Total	5 day Moving Average
1	26	–	–
2	29	–	–
3	35	188	37.600
4	47	188	37.600
5	51	191	38.200
6	26	193	38.600
7	32	192	38.400
8	37	194	38.800
9	46	196	39.200
10	53	194	38.800
11	28	193	38.600
12	30	193	38.600
13	36	194	38.800
14	46	194	38.800
15	54	195	39.000
16	28	195	39.000
17	31	195	39.000
18	36	195	39.000
19	46	–	–
20	54	–	–

## Exercise:

## Theoretical Questions

⊙ Multiple Choice Questions (MCQ)

1. By using arithmetic mean method the index number from the following data is

Commodity	Base price	Current price	Weight
Rice	30	52	8
Wheat	25	30	6
Fish	130	150	3
Potato	35	49	5
Oil	70	105	7

- (a) 144.92  
 (b) 202.34  
 (c) 161.87  
 (d) 115.22
2. From the data given below the wholesale price index number for the year 1 taking year 0 as base using simple arithmetic average of relatives method is

Commodity	Price year 0	Price year 1
A	80	120
B	120	150
C	40	80
D	100	150
E	200	240

- (a) 180  
 (b) 112  
 (c) 134  
 (d) 149
3. Consider the following:

Year	1	2	3	4	5	6	7	8
Annual Sales (₹ '0000)	3.6	4.3	4.3	3.4	4.4	5.4	3.4	2.4

5 year moving average against year 4 is

- (a) 3.80  
 (b) 4.00

- (c) 4.36  
(d) 4.18

4. Consider the following:

Year	1	2	3	4	5	6	7	8
Annual Sales (₹'0000)	3.6	4.3	4.3	3.4	4.4	5.4	3.4	2.4

4 year centered moving average against year 6 is

- (a) 4.00  
(b) 4.24  
(c) 4.26  
(d) 4.025
5. Consider the following table:

Item	Year 1		Year 2	
	Unit Price	Quantity	Unit Price	Quantity
I	1	16	3	15
II	3	15	8	20
III	5	18	10	21

Pasche's price index of year 2 with respect to year 1 is

- (a) 189.13  
(b) 230.56  
(c) 245.12  
(d) 256.78
6. Fisher's ideal index for prices from the following data is

Item	Base Year		Current Year	
	Unit Price	Quantity	Unit Price	Quantity
A	8	6	12	5
B	10	5	11	6
C	15	8	10	5

- (a) 97.72  
(b) 80.15  
(c) 95.67  
(d) 89.14

7. Consider the following series of observation.

Year	1	2	3	4	5	6	7	8	9	10	11
Sales (₹)	2	6	1	5	3	7	2	6	4	8	3

4 year centered moving average against year 6 is

- (a) 5.125
- (b) 3.875
- (c) 3.625
- (d) 4.375

8. Consider the following series of observation.

Year	1	2	3	4	5	6	7	8	9	10	11
Sales (₹)	2	6	1	5	3	7	2	6	4	8	3

5 year weighted moving average with weights 1, 2, 2, 2, 1 against year 6 is

- (a) 5.125
- (b) 3.875
- (c) 3.625
- (d) 4.375

9. Consider the following series of observation.

Year	1	2	3	4	5	6	7	8	9	10	11
Sales (₹)	2	6	1	5	3	7	2	6	4	8	3

5 year moving average against year 6 is

- (a) 3.6
- (b) 4.6
- (c) 4.4
- (d) 5.4

10. Consider the following:

Commodity	Base Price (₹)	Current price (₹)	Weight
A	22	40	8
B	15	15	6
C	80	90	7
D	110	130	3
E	25	30	5

Weighted aggregative index number is

- (a) 123.34
- (b) 156.11
- (c) 176.52
- (d) 142.89

11. Consider the following:

Commodity	Base Price (₹)	Current price (₹)	Weight
A	22	40	8
B	15	15	6
C	80	90	7
D	110	130	3
E	25	30	5

Weighted A.M of price relative index number is

- (a) 123.34
- (b) 128.79
- (c) 130.92
- (d) 182.13

12. From the following find the Fisher's Quantity index

Item	Base Year		Current Year	
	Unit Price ( $P_0$ )	Quantity ( $Q_0$ )	Unit Price ( $P_1$ )	Quantity ( $Q_1$ )
A	8	6	12	5
B	10	5	11	6
C	17	8	8	5

- (a) 32.76
- (b) 72.34
- (c) 82.89
- (d) 12.74

13. From the following find the Simple average (AM) of Relative Quantity index

Item	Base Year Quantity	Current Year Quantity
A	8	12
B	10	11
C	15	10

- (a) 111.45
- (b) 108.89
- (c) 32.45
- (d) 115.46

14. From the following find the Simple average (GM) of Relative Quantity index

Item	Base Year Quantity	Current Year Quantity
A	8	12
B	10	11
C	15	10

- (a) 100.23  
 (b) 111.45  
 (c) 190.15  
 (d) 103.23

15. From the following data the five year moving average against year 5

Year	1	2	3	4	5	6	7	8	9
Sales (₹)	36	43	43	34	44	54	34	24	14

- (a) 40  
 (b) 43.6  
 (c) 34  
 (d) 41.8

16. From the following data the four year centered moving average against year 6

Year	1	2	3	4	5	6	7	8	9
Sales (₹)	36	43	43	34	44	54	34	24	14

- (a) 40.25  
 (b) 40.625  
 (c) 35.25  
 (d) 40

17. From the following series find out a three year moving average against year 4 is

Year	1	2	3	4	5	6	7
Values	12	14	15	17	18	20	23

- (a) 20.33  
 (b) 18.33  
 (c) 16.67  
 (d) 15.33

18. From the following series find out a three year weighted moving average against year 4 with weights 1, 4, 1 is

Year	1	2	3	4	5	6	7
Values	12	14	15	17	18	20	23

- (a) 20.17  
 (b) 16.83  
 (c) 18.17  
 (d) 15.17

19. Consider the following table:

Commodity	Weights	Base price p.u. (₹)	Current price p.u (₹)
A	40	16	30
B	25	40	70
C	5	0.5	1.5
D	20	5.12	7.25
E	10	2	2.5

Weighted A.M price relative index is

- (a) 146.98  
 (b) 174.57  
 (c) 124.33  
 (d) 156.01

20. From the following four year centered moving average against year 4 is

Year	1	2	3	4	5	6	7
Import (₹ m)	229	231	206	191	195	184	193

- (a) 190.671  
 (b) 199.875  
 (c) 192.375  
 (d) 210

**Answer:**

⊙ **Multiple Choice Questions (MCQ)**

1.	2.	3.	4.	5.	6.	7.	8.	9	10.
(a)	(d)	(c)	(d)	(b)	(a)	(d)	(d)	(b)	(a)
11.	12.	13.	14.	15.	16.	17.	18.	19	20.
(c)	(c)	(b)	(d)	(d)	(a)	(c)	(b)	(b)	(b)

