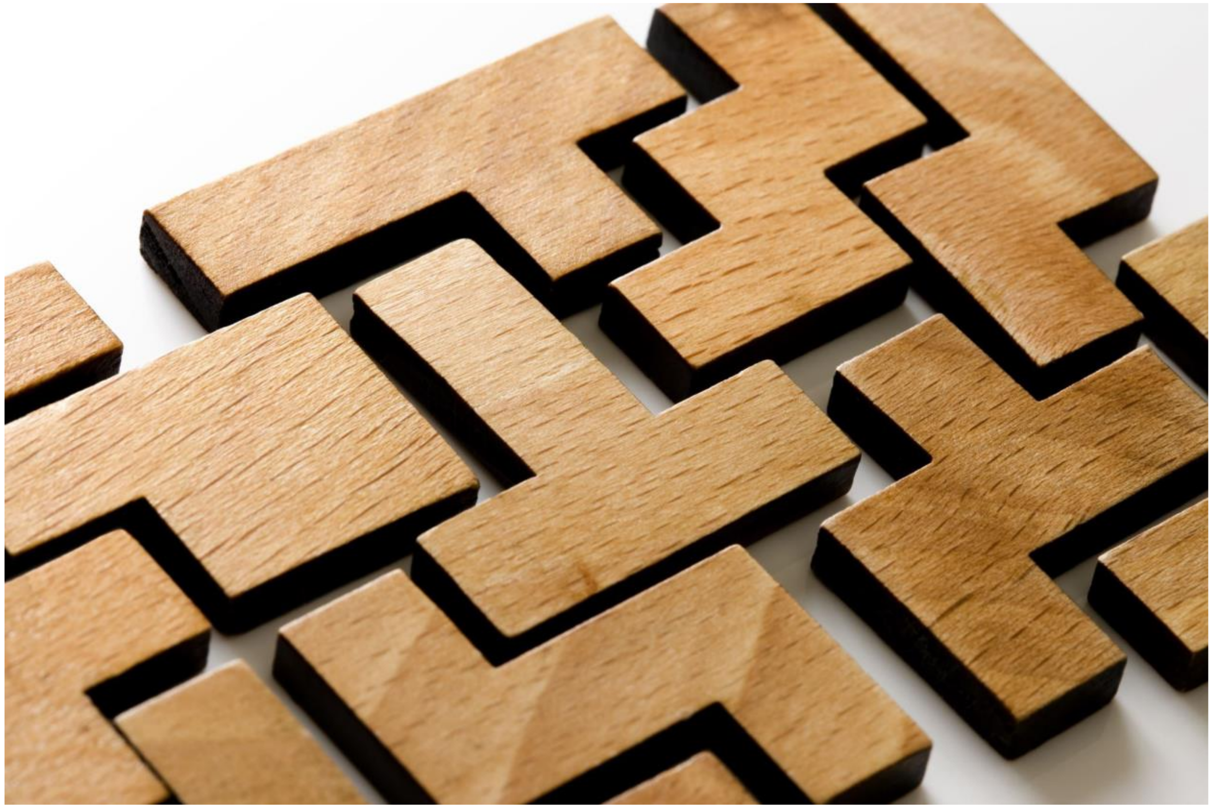

Time Value of Money

Compounding and Discounting of Cash Flows

An Example

- Suppose Shashi owns a house in the Nilgiris
- She is not using the house and hence decides to sell it
- Ramesh agrees to buy the house at Rs. 10 lakhs
  - Ramesh will pay the money on signing of the contract
- Rohan is also willing to buy the house at Rs. 11.5 lakhs
  - Rohan will pay the entire money after 1 year from the date of signing the contract
- What should Shashi do?
  - Shashi banks with Canara Bank and Yes Bank
  - FD rates for Canara Bank and Yes Bank for an FD maturing by or in 24 months are 7% p.a. and 9% p.a. respectively, interest being applied annually

The Story

- Let, $t = 0$, represent the date of signing the contract
  - $\therefore t = 1\ represents\ 1\ year\ from, t = 0$

t	Cash inflow to Sashi (Rs. Lakhs)
0	10
1	11.5

- Can we compare?
- What can Sashi do if she sells to Ramesh?
  - She can park her receipts in an FD with either of the Bank
  - Calculate, what it will be at $t = 1$

The Story…

- If she invests in a FD with Canara Bank for a period of 12 months,
  - $Value\ of\ the\ FD\ (at\ t = 1) = 10 * (1 + 0.07) = 10.7$
- Similarly with Yes Bank,
  - $Value\ of\ the\ FD\ (at\ t = 1) = 10 * (1 + 0.09) = 10.9$

At, t=1	Cash inflow (Rs. Lakhs)
FD at Canara Bank	10.7
FD at YES Bank	10.9
Payment by Rohan	11.5*

An Example…what if…

- Suppose Shashi owns a house in the Nilgiris
- She is not using the house and hence decides to sell it
- Ramesh agrees to buy the house at Rs. 10 lakhs
    - Ramesh will pay the money on signing of the contract
- Rohan is also willing to buy the house at Rs. 11.5 lakhs
    - Rohan will pay the entire money after 2 years from the date of signing the contract
- What should Shashi do?
    - Shashi banks with Canara Bank and Yes Bank
    - FD rates for Canara Bank and Yes Bank for an FD maturing by or in 24 months are 7% p.a. and 9% p.a. respectively, interest being applied annually

The 'what if…' Story

- Let, $t = 0$, represent the date of signing the contract
  - $\therefore t = 1\, represents\, 1\, year\, from, t = 0$

t	Cash inflow to Sashi (Rs. Lakhs)
0	10
2	11.5

- Can we compare?
- What can Sashi do if she sells to Ramesh?
  - She can park her receipts in an FD with either of the Banks
  - Calculate, what it will be at $t = 2$

The 'what if…'Story…

- If she invests in a FD with Canara Bank for a period of 24 months,
  - $Value\ of\ the\ FD\ (at\ t = 1) = 10 * (1 + 0.07) = 10.7$
  - $Value\ of\ the\ FD\ (at\ t = 2) = 10 * (1 + 0.07) * (1 + 0.07) = 10 * (1 + 0.07)^2 = 11.449 \approx 11.45$
- Similarly with Yes Bank,
  - $Value\ of\ the\ FD\ (at\ t = 2) = 10 * (1 + 0.07) * (1 + 0.07) = 10 * (1 + 0.07)^2 = 11.881 \approx 11.88$

At, t=2	Cash inflow (Rs. Lakhs)
FD at Canara Bank	11.45
FD at YES Bank	11.88*
Payment by Rohan	11.5

So what did we learn?

- $FV = PV(1 + r)^n, where$
  - $FV = Future\ Value$
  - $PV = Present\ Value$
  - $r = rate\ of\ interest$
  - $n = period\ of\ investment$
- This process is called COMPOUNDING

Problem: Semi-annual Interest application

- Rohit has generated a surplus fund of Rs. 12 Lakhs. He has deposited the sum in a FD with Radian Bank for a period of 5 years @ 8% p.a., interest being applied semi-annually
- How much does Rohit expect to receive at the end of 5 years?
  - Compare the result with the case when interest is applied annually

- Semi-annual application of interest is same as Half-yearly application
  - Frequency of compounding is 2 times per year
    - Once, every 6 months

Problem…

- In the case of semi-annual interest,
  - $FV = PV * (1 + \frac{r}{2})^{(2n)}$
- Here,
  - $FV = 12 * (1 + \frac{0.08}{2})^{(2*5)} = 17.76$
- If interest is charged annually, $FV = 17.63$

Let us apply what we learned

- Suppose Ismat has invested Rs. 50 lakhs in an FD with Radian Bank for a period of 5 years. At the end of 5 years, she expects to get Rs. 62 lakhs. What is the rate of interest being offered by the bank (assume no taxes)?
  - $FV = PV(1+r)^n$

  i.e., $(1+r)^n = \frac{FV}{PV}$

  i.e., $(1+r) = (\frac{FV}{PV})^{\frac{1}{n}}$

  i.e., $r = (\frac{FV}{PV})^{\frac{1}{n}} - 1$

  - $Answer: r = 4.4\%\ p.a.$*

The story of discounting

- Consider this example:
  - Muskaan is planning to buy a Macbook Air after 12 months.
  - The estimated price of Macbook Air at the end of 12 months is Rs. 1.00 Lakhs
  - She is planning to deposit some money (from a surplus – she has now) in an FD with Canara Bank so that after 12 months she has just enough funds for Macbook Air
    - She wants to know how much she should deposit now.
    - She knows that at present Canara Bank is paying 7.5% p.a. at annual rest for FDs below Rs. 2.00 Lakhs

The story…

- Assume, $x = Money\ that\ Muskan\ must\ deposit\ now$
- We know:
  - $x(1 + 0.075) = 1 \rightarrow x = \frac{1}{(1+0.075)} = 0.93$
  - Muskan must deposit Rs. 0.93 Lakhs now so that she can get Rs. 1.00 Lakhs after 1 year

- $Note: FV = PV(1 + r)^n \rightarrow PV = \frac{FV}{(1+r)^n}$
  - This process is called DISCOUNTING

Another example

- Rishiraj is planning to pursue an MBA course after 5 years (i.e. 2025)
  - The expected cost of this course is Rs. 30.0 Lakhs
  - Rishiraj's parents are planning for their son's education
  - One strategy that his parents are considering is:
    - Deposit a lumpsum amount now as FD with Axis Bank for a period of 5 years, that will grow to Rs. 30.0 Lakhs at maturity
    - If Axis Bank is paying 8% p.a. at annual rest, how much should they deposit now?

Example… contd.

- $30 = PV(1 + 0.08)^5 \rightarrow PV = \frac{30}{(1+0.08)^5} = 20.42$
- When his parents visit the bank, the dealing officer announces that they may choose interest (8% p.a.) application at either annual rest, semi-annual rests, quarterly rests or monthly rests
  - Can you advise Rishi's parents about amounts they should deposit under various scenarios?
  - Which scenario they should choose?

Example…Contd.

Scenario	Formula	Amount of Deposit (Rs. Lakhs)
Interest application: Annual	$PV = \frac{TV(=30)}{(1+0.08)^5}$	20.42
Interest application: Semi-annual	$PV = \frac{TV}{(1+\frac{0.08}{2})^{10}}$	20.27
Interest application: Quarterly	$PV = \frac{TV}{(1+\frac{0.08}{4})^{20}}$	20.19
Interest application: Monthly	$PV = \frac{TV}{(1+\frac{0.08}{12})^{60}}$	20.13

Stated Rate of Interest and Effective Rate of Interest

- Think of Rishiraj story
  - The stated annual rate of interest is the annual rate of interest without considering the frequency of compounding
  - The Effective Rate of Interest, is the rate of interest arrived at after considering the frequency of compounding

- $Let\ m = frequency\ of\ compounding\ per\ year$
  - $Effective\ Rate\ of\ Interest = (1 + \frac{r}{m})^{m} - 1, where, r = stated\ rate\ of\ interest$

Effective Rate of Interest: Rishiraj Case

Scenario	Stated Rate of interest	Frequency of Compounding (m)	Effective Rate of Interest (% p.a.)
Interest application: Annual	8% p.a.	1	$[(1+\frac{0.08}{1})^{1}]$-1=8.0%
Interest application: Semi-annual	8% p.a.	2	$[(1+\frac{0.08}{2})^{2}]$-1=8.16%
Interest application: Quarterly	8% p.a.	4	$[(1+\frac{0.08}{4})^{4}]$-1=8.24%
Interest application: Monthly	8% p.a.	12	$[(1+\frac{0.08}{12})^{12}]$-1=8.29%

FVIF & PVIF

- The factor: $(1+r)^n$ is called Future Value Interest Factor ($FVIF_{r,n}$)
- The factor: $\frac{1}{(1+r)^n}$ is called Present Value Interest Factor ($PVIF_{r,n}$)

  - Value FVIF and PVIF depends on
    - Rate of interest (r)
    - Period of Investment (n)

Table for PVIF

■ **TABLE A.1** **Present Value of \$1 to Be Received after T Periods = $1/(1 + r)^T$**

	Interest Rate								
Period	**1%**	**2%**	**3%**	**4%**	**5%**	**6%**	**7%**	**8%**	**9%**
1	0.9901	0.9804	0.9709	0.9615	0.9524	0.9434	0.9346	0.9259	0.9174
2	0.9803	0.9612	0.9426	0.9246	0.9070	0.8900	0.8734	0.8573	0.8417
3	0.9706	0.9423	0.9151	0.8890	0.8638	0.8396	0.8163	0.7938	0.7722
4	0.9610	0.9238	0.8885	0.8548	0.8227	0.7921	0.7629	0.7350	0.7084
5	0.9515	0.9057	0.8626	0.8219	0.7835	0.7473	0.7130	0.6806	0.6499
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

- Similar table is available for FVIF

Problem

- Sushant has just won a lottery and has received Rs. 10 Lakhs
  - He wants to buy a car in 5 years
    - The expected price of the car is Rs. 16.10 Lakhs
  - At what interest rate must he invest his proceeds so that he can afford the car in 5 years?

- Answer: $9.99\%\ p.a. \approx 10\%\ p.a.$

Example: Radian Systems

- Radian Systems is considering investment in a high-speed server
  - The server costs Rs. 50 Lakhs
- The server will enable Radian to generate revenue as per the following schedule
  - Rs. 25.00 Lakhs – in Year 1
  - Rs. 20.00 Lakhs – in Year 2
  - Rs. 15.00 Lakhs – in Year 3
- After 3 years, the Server will be rendered useless
  - However, the scrap value is estimated as Rs. 2 lakhs
- Finance team of Radian applies 7% p.a. as discount factor for all its investments

- Should Radian invest in the Server?

Radian Systems…Contd.

Time	Cashflow for Radian (Rs. Lakhs)	Remarks
0	-50	Cash outflow
1	+25	Cash Inflow
2	+20	Cash Inflow
3	+15+2 =+17	Salvage Value is also a Cash inflow

Radian Systems…Contd.

- $Net\ Present\ Value\ (NPV) = -C_0 + \sum_{t=1}^{n} \frac{C_t}{(1+r)^t}$
- $Here, NPV = -50 + \frac{25}{(1+0.07)^1} + \frac{20}{(1+0.07)^2} + \frac{17}{(1+0.07)^3} = 4.71$

- What does this mean?
  - The PV of the Revenue Streams exceeds Initial Investment by Rs. 4.71 Lakhs
  - The investment is rational decision

- Decision Rule:
  - Accept the Investment if NPV>0
  - Reject the Investment if NPV<0
  - Indifferent to the Investment if NPV=0

NPV: Some important issues

- Effectively all investments decisions
  - Financial
  - Corporate investment is Capital Goods
  - Government investment in Infrastructure
  - Many others
- Note, $NPV \propto \frac{1}{r}\ or, \frac{d(NPV)}{dr} < 0$
- In case of certain projects, the initial investment is spread over multiple periods
  - Gestation period may be > 1 year
  - The period during which the project is set up, installed and commissioned is called gestation period
    - Gestation periods are characterized by Net Cashflow <0
  - In such cases, $NPV = \sum_{t=0}^{n} \frac{C_t}{(1+r)^t}, as\ C_0 = \frac{C_0}{(1+r)^0}$

Stated Numbers can be Misleading

- A Newspaper report August 2019
  - *Virat Kohli has entered into a Rs. 55 Crores Contract with Nike. Virat will receive Rs. 6 Cr. as signing bonus plus Rs. 49 Cr. as endorsement fees over 5 years : Rs. 1 Cr. In 2019 and Rs. 12 Cr. Annually during 2020-2023.*

- Assuming that the rate of discount is 12% p.a., what is the actual value of Virat's Contract?

Explanation

Year	Time (t)	Cashflow to Kohli	PV (Cashflow) @12% p.a.
2019	0	6+1=7	7
2020	1	12	10.71
2021	2	12	9.57
2022	3	12	8.54
2023	4	12	7.63
Total			**43.44**

Continuous Compounding…What does it mean?

- When compounding happens very frequently
  - Say, at every minute
  - Or, say, at every second

- This phenomenon is called continuous compounding
- In the case of Continuous Compounding,
  - $TV = PV.e^{rt}$
  - Where, $r = Stated\ annual\ rate\ of\ interest; t = number\ of\ years\ over\ which\ the\ investment\ runs$

Proof: Continuous Compounding

- Let us assume,
  - $P_0 = Present\ Value\ of\ the\ Investment$
  - $r = Stated\ rate\ of\ Interest\ (expressed\ as\ \%\ p.a.)$
  - $m = Frequency\ of\ compounding\ per\ year$
  - $t = Number\ of\ years\ for\ which\ the\ investment\ is\ made$
- We know,
  - $TV = P_0(1 + \frac{r}{m})^{m.t} \Longrightarrow \frac{TV}{P_0} = (1 + \frac{r}{m})^{m.t}$

Proof…Contd.

- We try to find,
  - $\frac{TV}{P_0} = \lim_{m\to\infty}(1+\frac{r}{m})^{m.t}$
  - Now,
    - $\ln\left(\frac{TV}{P_0}\right) = \ln\left[\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{m.t}\right] \Rightarrow \ln\left(\frac{TV}{P_0}\right) = \lim_{m\to\infty}[\ln\left(1+\frac{r}{m}\right)^{m.t}]$….(1)
  - Now, $\ln\left(\left(1+\frac{r}{m}\right)^{m.t}\right) = mt.\ln(1+\frac{r}{n}) \dots.(1)$
  - $Let, \frac{r}{m} = h,$
  - $\therefore$ By Taylor's Series expansion: $\ln(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \cdots$

Proof…contd.

- Hence,
  - $\ln\left(1+\frac{r}{m}\right)=\frac{r}{m}-\frac{r^2}{2m^2}+\frac{r^3}{3m^3}-\frac{r^4}{4m^4}+\cdots$
  - $Or, mt.\ln\left(1+\frac{r}{m}\right)=t\left[r-\frac{r^2}{2m}+\frac{r^3}{3m^2}-\frac{r^4}{4m^3}+\cdots\right]$

$$\therefore \lim_{m\to\infty}\left[mt.\ln\left(1+\frac{r}{m}\right)\right]=rt$$

  - From (1), $\ln\left(\frac{TV}{P_0}\right)=rt \Rightarrow \frac{TV}{P_0}=e^{rt} \Rightarrow TV=P_0e^{rt}=PV.e^{rt}$

- Note that in case of Discounting,
  - $PV=TV.e^{-rt}$

A problem: Continuous Compounding

- Aastha decides to invest Rs. 10.0 Lakhs for 2 years in an instrument that yields 9% p.a. with continuous compounding.
  - How much will Aastha receive at the end of maturity of her investment

- Ans: Rs. 11.97 Lakhs

Solution

- $PV = 10, t = 2, r = 9\%\, p.a.$
  - $TV = 10.e^{(2*0.09)} = 11.97$

- Note: in Excel, $e^{x}\ is\ found\ as[= \exp(x)]$

A Problem: Knox Capital

- Knox Capital is accepting deposits, that promises an investor to pay an interest of 8% p.a. The investor may choose either (a) Quarterly compounding, (b)Monthly compounding, (c) Continuous compounding. The period of deposits is 5 years
  - An investor wishes to receive Rs. 25 Lakhs at the end of 5 years
  - Which scheme will she choose? How much should she invest now?

Knox Capital...Contd.

TV (Rs. Lakhs)	Rate of interest	Frequency of Compounding	PV (Rs. Lakhs)
25	8% p.a.	Quarterly	$\frac{25}{(1+\frac{0.08}{4})^{(4x5)}} = 16.82$
25	8% p.a.	Monthly	$\frac{25}{(1+\frac{0.08}{12})^{(12x5)}} = 16.78$
25	8% p.a.	Continuous	$25.e^{-(0.08x5)} = 16.75$

Annuity

- What is an Annuity?
  - When cashflow for each period is a fixed (constant amount) and the cashflows last for a fixed length of time, the cashflow is called an Annuity

- $When\ C_1 = C_2 = C_3 = \cdots = C_n = C(say)$, the cash flow is called annuity
  - $Here, n\ is\ a\ finite\ number$

- Annuity is one of the most common type of cashflow associated with financial instruments
  - Many types of Life Insurance Premium
  - EMI of loans
  - Lease Rental Payments

PV of an Annuity

- $Let\ the\ annuity\ be$:
  - Cashflow per period = C
  - Tenor of the annuity = n
  - Rate of Discount= r% p.a. with annual compounding
- Let the PV of the Annuity =PV_A
- $PV_A = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^n} = \sum_{t=1}^{n} \frac{C}{(1+r)^t}$
  - Our task is to find $the\ value\ of\ PV_A$

Deriving the Result

- $Let, \frac{1}{(1+r)} = x \;\Rightarrow PV_A = C[x + x^2 + x^3 + \cdots + x^n] \ldots\ldots\ldots\ldots$ (1)
- $Multiplying\ both\ sides\ of\ (1), we\ get,$
  - $xPV_A = C[x^2 + x^3 + x^4 + \cdots + x^{(n+1)}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$(2)
- $(1-2) \Longrightarrow (PV_A - xPV_A) = C[x - x^{(n+1)}]$
- $Or, PV_A(1-x) = Cx(1-x^n) \Longrightarrow PV_A = \frac{Cx(1-x^n)}{(1-x)} \ldots\ldots\ldots\ldots\ldots$(3)
- Substituting $x = \frac{1}{(1+r)}\ in\ (3), we\ get,$
- $PV_A = \frac{\frac{C}{(1+r)}[1-\frac{1}{(1+r)^n}]}{[1-\frac{1}{(1+r)}]} = \frac{C}{r}\left[1 - \frac{1}{(1+r)^n}\right] = C[\frac{1}{r} - \frac{1}{r(1+r)^n}] \ldots\ldots\ldots\ldots$(Ans.)
- Note:$\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right]$ $is\ called\ the\ Present\ Value\ Interest\ Factor\ of\ Annuity(PVIFA_{(r,n)})$

Example: Million Rupees Lottery

- Rupal has won the Million Rupees Lottery. She will get Rs. 50000 per year for the next 20 years (Rs. 50000 x 20 = Rs. 1 Million). Rupal uses 8% p.a. for discounting her cashflows. What is the PV of the proceeds of the Million Dollar Lottery to Rupal?
  - Should Rupal rejoice?

- Rupal's Present Value Annuity factor is
  - $\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right] = \left[\frac{1}{0.08} - \frac{1}{0.08(1+0.08)^{20}}\right] = 9.8181$

- PV of the cashflow to Rupal is: $Rs. 50000 * 9.8181 = Rs. 490907 \approx Rs. 4.91\ Lakhs$
  - Rupal should actually file a legal case!!!!!!

Annuity Future Value: Example of a Pension Plan

- Sushmita has planned for her retirement. She has opted for a Pension scheme of LIC wherein she will deposit Rs. 10500 for each year for 30 years. The scheme will pay interest at 9% p.a. What is the amount she will receive after 30 years?

  - LIC will pay her an annual pension of uniform amount for 20 years starting from the 31^{st} year and using a discount rate of 10.5% p.a.

  - How much will Sushmita receive as Yearly Pension, staring from the 31^{st} Year to 50^{th} Year?

Problem: Pension Plan - Explained

- $Let\ the\ annual\ contribution = A\ (in\ Shusmita's\ Case, A = 10500)$
  - The amount deposited in Year 1 will grow to $A(1+r)^{(n-1)}$
  - Amount deposited in Year 2 will grow to $A(1+r)^{(n-2)}$
  - Amount deposited in Year 3 will grow to $A(1+r)^{(n-3)}$
  - …….
  - ……
  - Amount deposited in Year (n-2) will grow to $A(1+r)^2$
  - Amount deposited in Year (n-1) will grow to $A(1+r)$
  - Amount deposited in the n^{th} year = A
- Let, $The\ Future\ Value\ of\ the\ Annuity = FV_A$
  - $FV_A = A[(1+r)^{(n-1)} + (1+r)^{(n-2)} + (1+r)^{(n-3)} + \cdots + (1+r)^2 + (1+r) + 1]$

Pension Plan Explained…contd.

- $Let, (1+r) = y$
- $\therefore\ FV_A = A[y^{(n-1)} + y^{(n-2)} + y^{(n-3)} + \cdots + y^2 + y + 1]$………(1)
- $Multiplying\ both\ sides\ by\ y, we\ get,$
  - $yFV_A = A[y^n + y^{(n-1)} + y^{(n-2)} + \cdots + y^3 + y^2 + y]$……………..(2)
- $(2) - (1) \Longrightarrow FV_A(y-1) = A[y^n - 1]$
- $Or, FV_A = \frac{A[y^n - 1]}{(y-1)}$
- $Substituting\ the\ value\ of\ y = (1+r), we\ get,$
  - $FV_A = \frac{A[(1+r)^n - 1}{r} = A[\frac{(1+r)^n}{r} - \frac{1}{r}]$
  - $The\ Factor, \left[\frac{(1+r)^n}{r} - \frac{1}{r}\right] is\ called\ Future\ Value\ Interest\ Factor\ of\ Annuity\ [FVIFA_{(r,n)}]$

Pension Plan Explained…Contd.

- In the case of Sushmita,
  - $n = 30, r = 0.09, \mathrm{A} = 10500$
  - $\therefore FVIFA = \left[\frac{(1+0.09)^{30}}{0.09} - \frac{1}{0.09}\right] = 136.3075$
  - $Hence, the\ future\ value\ of\ the\ Annuity = A.FVIFA = 10500 * 136.3075 \approx 1431229$
- At the end of 30 years, for Sushmita, there is a corpus of Rs. 1431229, which will be distributed as an annual pension (of equal amount over the next 20 years

Pension Plan Explained…Contd.

- $Let\ the\ annual\ Pension = P$
  - $n = 20; r = 10.5\%\ p.a.$
- Now, $PV(Annuity) = C\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right], with\ symbols\ having\ usual\ meanings$
- Here, according to the Problem,
  - $1431229 = P\left[\frac{1}{0.1050} - \frac{1}{0.1050(1+0.1050)^{20}}\right] = P * 8.2309$
  - $Or, P = \frac{1431229}{8.2309} \cong 173884$

Growing Annuity

- What is a growing annuity?
    - A finite number of growing cash flows, with uniform rate of growth over the tenor
- $Let, the\ payment\ in\ Year\ 1 = C$
    - $Payment\ in\ Year\ 2 = C(1+g), where, 0 < g < 1$
    - $Payment\ in\ Year\ 3 = C(1+g)(1+g) = C(1+g)^2$
    - $\ldots\ldots.$
    - $Payment\ in\ Year\ (n-1) = C(1+g)^{(n-2)}$
    - $Payment\ in\ Year\ n = C(1+g)^{(n-1)}$
- $Discounting\ rate = r\ such\ that\ r \neq g$

PV of a Growing Annuity: Derivation

- $PV_{GA} = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots + \frac{C(1+g)^{(n-2)}}{(1+r)^{(n-1)}} + \frac{C(1+g)^{(n-1)}}{(1+r)^n}$
- $Or, PV_{GA} = \frac{C}{(1+r)}[1 + \frac{(1+g)}{(1+r)} + \frac{(1+g)^2}{(1+r)^2} + \cdots + \frac{(1+g)^{(n-2)}}{(1+r)^{(n-2)}} + \frac{(1+g)^{(n-1)}}{(1+r)^{(n-1)}}]$
- $Or, PV_{GA} = \frac{C}{(1+r)}[1 + x + x^2 + \cdots + x^{(n-2)} + x^{(n-1)}], where, x = \frac{(1+g)}{(1+r)}$..............................(1)
- $Multiplying\ both\ sides\ of\ (1) by\ x, we\ get,$
  - $x.PV_{GA} = \frac{C}{(1+r)}[x + x^2 + x^3 + \cdots + x^{(n-1)} + x^n]$...(2)
- (1-2)$\Longrightarrow (1-x)PV_{GA} = \frac{C}{(1+r)}[1 - x^n]$
- Substituting x, we get,
- $\left[1 - \frac{(1+g)}{(1+r)}\right] PV_{GA} = \frac{C}{(1+r)}\left[1 - \left(\frac{(1+g)}{(1+r)}\right)^n\right] \Longrightarrow (r-g)PV_{GA} = C[1 - \left(\frac{(1+g)}{(1+r)}\right)^n]$
- Or, $PV_{GA} = C[\frac{1-(\frac{(1+g)}{(1+r))})^n}{(r-g)}]$.........(Proved)

Problem: The story of Munaf

- Our friend Munaf had applied for a job at Aditya Birla Capital. He has recently got an offer letter from ABCL.
  - He is expected to join ABCL as a Financial Analyst, soon after passing his 6^{th} Semester
  - His annual take-home salary is Rs. 6 Lakhs (net of Taxes and statutory payments)
  - If he continue to perform as per expectation of the Company, the annual take-home salary will increase by 9% p.a. every year
  - As of now Munaf expects to remain with ABCL for a period of 20 years
  - Munaf discounts his cash flows at 12% p.a.

  - What is the present value of Munaf's take-home salary from his career with ABCL?

Story of Munaf…Contd.

- In this case,
  - $C = 6; n = 20; g = 0.09; r = 0.12$

- Since this is case of growing annuity,
  - $PV_{GA} = \frac{C}{(r-g)}[1 - (\frac{(1+g)}{(1+r)})^n] \Longrightarrow \frac{6}{(0.12-0.09)}[1 - \left(\frac{(1+0.09)}{(1+0.12)}\right)^{20}\right] = 83.80 \text{ (Ans)}$

What is the FV of a Growing Annuity?

- Suppose,
  - $Numper\ of\ periods = n$
  - $First\ Deposit\ of\ an\ Annuity = C$
  - $Annual\ Growth\ rate\ of\ Annuity = g, with\ 0 < g < 1$
  - $Rate\ of\ Discounting = r$
  - $Let, the\ Future\ Value\ of\ a\ growing\ annuity = FV_{GA}$

- $Now, we\ know, FV_{GA} = PV_{GA}(1+r)^n \Longrightarrow FV_{GA} = C[\frac{1-(\frac{(1+g)}{(1+r))})^n}{(r-g)}](1+r)^n$

- $Or, FV_{GA} = \frac{C}{(r-g)}\left[(1+r)^n - \frac{(1+g)^n}{(1+r)^n}.(1+r)^n\right] = \frac{C}{(r-g)}[(1+r)^n - (1+g)^n]$

Problem: Medha's Savings Plan

- In August 2020, Medha is wondering about her future. She calculates that as on date, she has about 25 years of service life left, before she retires
- She wishes to have Rs. 10 Cr. as Savings when she retires.
- She decides to save in a fashion, such that, at the end of her service life, she has the desired savings
  - She will start saving from next year (2021)
  - She also decides that every year following 2021, she will increase her annual savings by 10% compared to the previous year
- Medha discounts her cash flows by 12%.
- How much will Medha save in 2021

Medha's Savings Plan…Contd.

- For Medha, we have the following data:
  - $FV\ of\ Savings = 10\ Cr.$
  - $g = 10\%$
  - $r = 12\%$
  - $n = 25$
  - $Let, her\ initial\ savings\ (in\ 2021) = C$

- $FV_{GA} = \frac{C}{(0.12-0.10)}[(1+0.12)^{25} - (1+0.10)^{25}] = 308.2679$
- $Or, 10 = 308.2679 * C \Longrightarrow C = \frac{10}{308.2679} = 0.0324\ Crores \cong 3.24\ Lakhs$

Perpetuity

- What is a Perpetuity?
  - A constant stream of cash flows without end
  - A constant stream of Cash flows (say, C per period) is paid in periods: 1, 2, 3,…………
  - Perpetuity comes from the fact that the cash flows are perpetual

- Examples
  - Since a Company is a ”going concern”, it's cash flows are Perpetuity
  - Certain Sovereign Bonds provide perpetual cashflows
  - Certain life insurance policies pay perpetual pensions (the period is "till death of the holder" – since the date of death is uncertain, it considered to be perpetuity)

PV of a Perpetuity

- $Let\ the\ payment\ per\ period = C$
- $Let, the\ discount\ rate = r\%\ p.a.$
- $Let, PV\ of\ the\ Perpetuity = \ PV_P$
- $PV_P = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$ ----------------------(1)
- $Multiplying\ both\ sides\ by\ \frac{1}{(1+r)}, we\ get,$
    - $\frac{1}{(1+r)} PV_P = \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \frac{C}{(1+r)^4} + \cdots$ -------------------(2)
- $(2-1) \Longrightarrow \left[1 - \frac{1}{(1+r)}\right] PV_P = \frac{C}{(1+r)} \Longrightarrow \frac{r.PV_P}{(1+r)} = \frac{C}{(1+r)} \Longrightarrow PV_P = \frac{C}{r}$

Problem: Valuing a Perpetual Sovereign Bond

- The Government of Botswana has floated a Perpetual Sovereign Bond. The structure of the Bond is as follows:
  - For the first 5 Years, the Bond will not pay any amount
  - Starting from Year 6, the Bond will pay US$ 1000 every year perpetually
  - The Bond is being sold at a Price of US$ 7500

- Mr. Dhiraj Roy is considering investing in this bond. Knowing that you are learning FinEco, he has sought your advice.
  - Dhiraj's discount rate is 8% p.a.

- What is your advice to Mr. Dhiraj Roy?

Problem: Botswana Bond

- $Annual\ Receipt\ starting\ from\ Year\ 6 = 1000$
    - $r = 0.08$

- $Value\ of\ the\ Perpetual\ Bond\ at\ the\ Start\ of\ Year\ 6 = \frac{1000}{0.08} = 12500$
    - Note "Start of Year 6" is equivalent to "End of Year 5"
- US$ 12500 is the value of the Cashflows from the Bond at the end of year 5
- $\therefore Value\ of\ the\ Bond\ Today = \frac{12500}{(1+0.08)^5} = 8507$
- Since Price of the Bond (US$ 7500) < Value of the Bond,
    - Mr. Roy may purchase the Bond, if he wishes
    - There is an Economic Rationale

Growing Perpetuity

- When a perpetuity grows @ g% p.a., it is called a growing perpetuity
- In Finance there are many applications of growing perpetuities
  - Valuation of Assets
  - Valuation of Firms
  - Valuation of Bonds,
  - Lease Rental Securitization, etc.

PV of a Growing Perpetuity

- $Let\ the\ first\ payment\ of\ a\ Perpetuity = C\ (in\ Year\ 1)$
  - Thereafter in each year, the payment grows by g% p.a.
  - $In\ Year\ 2 = C(1+g)$
  - $In\ Year\ 3 = C(1+g)(1+g) = C(1+g)^2$
  - ……………..
- $Let, PV\ of\ Cashflows = PV_{GP}; Rate\ for\ Discounting = r$
- $PV_{GP} = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots$ …………………(1)
- Multiplying both sides by: $\frac{(1+g)}{(1+r)}$,
- $\frac{(1+g)}{(1+r)} P_{GP} = \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \frac{C(1+g)^3}{(1+r)^4} + \cdots$ ----------------(2)
- $(1-2) \Rightarrow \left[1 - \frac{(1+g)}{(1+r)}\right] PV_{GP} = \frac{C}{(1+r)} \Rightarrow (r-g)PV_{GP} = C \Rightarrow PV_{GP} = \frac{C}{(r-g)}$ ----------Solution

Problem: Sunrise Apartment

- Ms. Shanti Pednekar owns the Sunrise Apartment on Marine Drive
  - There are 8 flats in this G+4 building
  - All the flats are on rent – as Shanti prefers this arrangement
    - She will continue with this arrangement in future
  - Shanti stays at Bandra
  - Shanti is planning her financial position
    - The annual net rental income from Sunrise Apartment next year will be Rs. 4.5 Crores
      - This is net of all maintenance expenses
    - Thereafter the Net Annual Rental Income from Sunrise is expected to grow at 5% p.a.
- What is the Present Value of the Rental Income from Sunrise, if Shanti uses 11% p.a. as discounting rate?

Sunrise Apartment…Continued.

- $Here, C_1 = 4.5;\ C_2 = 4.5(1 + 0.05);\ C_3 = 4.5(1 + 0.05)^2 \dots \dots ..$
- $The\ cash\ flows\ can\ be\ considered\ as\ growing\ perpetuity, since\ the\ number\ of\ periods\ are\ indefinite$
- $r = 0.11$

- $PV = \frac{C}{(r-g)} = \frac{4.5}{(0.11-0.05)} = 75$

- The Present Value of Rental Income (as of today) is Rs. 75 Crores

Thank you