LINEAR PROGRAMMING

Assumptions under LP

(i) Objective function and every constraint are linear.
(ii) Decision variables are not negative.
(iii) Increase or decrease in the quantities of decision variables will affect the objective function and every constraint in the proportional way.
(iv) All parameters (all coefficients in the objective function and the constraints) are known with certainty.
(v) Decision variables can be fractions.
(vi) The products, labour efficiency, machines etc. are assumed to be identical.

Terminology and Requirements of Linear Programming:

Regardless of the way one defines linear programming, certain basic requirements which are given below are necessary before the technique can be employed for optimization problems.

(i) Decision variables and their relationship.
(ii) Well defined objective function.
(iii) Presence of constraints or restrictions.
(iv) Alternative courses of action.
(v) Non-negative restrictions
(vi) Linearity.
(vii) Finiteness.
(viii) Additivity.
(ix) Divisibility
(x) Deterministic.

(i) Decision variables and their relationship:

The decision activity variables refer to candidates (products, services, projects etc.) that are competiting with one another for sharing the given limited resources. These variables are usually inter-related in terms of utilization of resources and need simultaneous solutions. The relationship among these variables should be linear.

(ii) Well defined objective function:

A linear programming problem must have a clearly defined objective function to optimize which may be either to maximize contribution by utilizing available resources, or it may be to produce at the lowest possible cost by using a limited amount of productive factors. It should be expressed as a linear function of decision variables.

(iii) Presence of constraints or restrictions:

There must be limitations on resources (like production capacity, manpower, time, machines, markets etc.) which are to be allocated among various competing activities. These must be capable of being expressed as linear equalities or inequalities in terms of decision variables.

(iv) Alternative courses of action:

There must be alternative courses of action. For example, it must be possible to make a selection between various combinations of the productive factors such as men, machines, materials, markets etc.

(v) Non-negative restrictions:

All decision variables must assume non-negative values as negative values of physical quantities is an impossible situation. If any of the variables is unrestricted in sign, a trick can be employed which enforces non-negativity changing the original information of the problem.

(vi) Linearity:

The basic requirement of a linear programming problem is that both the objective and constraints must be expressed in terms of linear equations or inequalities. It is well known that if the number of machines in a plant is increased, the production in the plant also proportionately increases. Such a relationship, giving corresponding increment in one variable for every increment in other, is called linear and can be graphically represented in the form of a straight line.

(vii) Finiteness:
There must be finite number of activities and constraints otherwise an optimal solution cannot be computed.

(viii) Additivity:
It means that sum of the resources used by different activities must be equal to the total quantity of resources used by each activity for all the resources individually and collectively. In other words, interaction among the activities of the resources does not exist.

(ix) Divisibility:
This implies that solutions need not be in whole numbers (integers). Instead, they are divisible and may take any fractional value. If a fraction of a product cannot be produced (like one fourth of a bus), an integer programming problem exists.

(x) Deterministic:
We assume that conditions of certainty exist i.e., the coefficients in the objective function and constraints are completely known (deterministic) and do not change during the period being studies e.g, profit per unit of each product, amounts of resources available are fixed during the planning period.

THREE PARTS OF LINEAR PROGRAMMING

We shall be studying this technique in three parts:

A. Problem formulation
B. Graphical Method of Linear Programming
C. Simplex Method of Linear Programming

A. PROBLEM FORMULATION

Problem formulation refers to translating the real life problem into a format of mathematical equalities and inequalities that abstracts all the essential elements of the problem. There are three parts of the formation (i) Objective function (ii) A set of constraints and (iii) Non-negativity restriction.

LP problem has got three points:

(i) Objective function: This describes the object of the management in precise and clear terms in quantitative form. We identify the decision variables and assume optimal values. For example if we have to find the optimal mix of two products, we may assume that we shall be producing x units of the first and y units of the second product. Suppose contribution per unit of the first product is ₹5 and the second one is ₹ 7, the objective function will be: Maximize $Z = 5x + 7y$.

(ii) A set of constraints: These are the limitations of the management expressed in quantitative form.

(iii) Non-negative restriction: This restriction prescribes that the decision variables should only be zero or positive.

Illustration 1:

A Mutual Fund has cash resources of ₹ 200 million for investment in a diversified portfolio. Table below shows the opportunities available, their estimated annual yields, risk factor and term period details.

Formulate a Linear Program Model to find the optimal portfolio that will maximize return, considering the following the following policy guidelines:

- All the funds available may be invested
- Weighted average period of at least five years as planning horizon.
- Weighted average risk factor not to exceed 0.20.
- Investment in real estate and speculative stocks to be not more than 25% of the monies invested in total.

Investment type	Annual yield (percentage)	Risk factor	Term period (years)
Bank deposit	9.5	0.02	6
Treasury notes	8.5	0.01	4
Corporate deposit	12.0	0.08	3
Blue-chip stock	15.0	0.25	5
Speculative stocks	32.5	0.45	3
Real estate	35.0	0.40	10

Solution:

Mathematical Formulation

Let x_1, x_2, x_3, x_4, x_5 and x_6 represent the six different investment alternatives, i.e., x_1 is bank deposit, x_2 is treasury note. X_3 corporate deposit, X_4 blue chip stock, x_5 speculative stock and x6 real estate. The objective is to maximize the annual yield of the investors (in number of units) given by the linear expression.

Maximize $Z = 9.5x_1 + 8.5x_2 + 12.0x_3 + 15.0x_4 + 32.5x_5 + 35.0x_6$

Subject to the constraints:

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 1$ (Investment decision)

$0.02x_2 + 0.01x_2 + 0.08x_3 + 0.25x_4 + 0.45x_5 + 0.40x_6 \leq 0.20$ (weighted average risk of the portfolio)

$6x_1 + 4x_2 + 3x_3 + 5x_4 + 3x_5 + 10x_6 \geq 5$ (weighted average length of investment)

$x_5 + x_6 \leq 0.25$ (limit on investment in real estate and speculated stock)

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$ (non-negativity condition)

Illustration 2:

A paper mill produces rolls of paper used in cash registers. Each roll of paper is 100 m in length and can be used in widths of 2, 4, 6 and 10 cm. The company's production process results in rolls that are 24 cm in width. Thus, the company must cut its 24 cm wide roll to the desired widths. It has six cutting alternatives as follows:

Cutting Alternative	Width of Rolls (cm.)				Waste (cm)
	2	4	6	10	
1	6	3	-	-	-
2	-	3	2	-	-
3	1	1	1	1	2
4	-	-	2	1	2
5	-	4	1	-	2
6	4	2	1	-	2

The minimum demand for the four rolls is as follows:

Roll (Width)	Demand
2	2,000
4	3,600
6	1,600
10	500

The paper mill wishes to minimize the waste resulting from trimming to size. **Formulate** the LP Model.

Solution:

Step 1. The key decision is to determine how the paper rolls be cut to the required width so that trim loss (wastage) is minimum.

Step 2. Let x_j (j= 1, 2................. 6) represent the number of times each cutting alternative is to be used. There alternatives result/do not result in certain trim loss.

Step 3. Feasible alternatives are sets of values of x_j
Where $x_j \geq 0$, j= 1,2…6

Step 4. The objective is to minimize the trim losses, i.e.

Minimize $Z = X_3 + X_4 + X_5 + X_6$

Formulation of LP Model

Objective Function: Minimize (wastage produced) $Z = 2(x_3 + x_4 + x_5 + x_6)$

Subject to the constraints

$6x_1 + X_3 + 4x_6 \geq 2{,}000$ … (For roll width of 2 cm.)

$3x_1 + 3x_2 + X_3 + 4X_5 + 2X_6 \geq 3{,}600$ … (For roll width of 4 cm.)

$2x_2 + X_3 + 2x_4 + X_5 + X_6 \geq 1{,}600$ … (For roll width of 6 cm.)

$X_3 + X_4 \geq 500$ …. (For roll width of 10 cm.)

$X_1, X_2, X_3, X_4, X_5, X_6 \geq 0$ … (Non negative)

B. Graphical Method

The following steps are considered:

(1) Formulate the appropriate Linear Programming Problem

(2) Construct graph for all the structural constraints. Convert each inequality into equality for each equation and select two points and plot them on the graph and connect by an appropriate line.

(3) After drawing the graph for each inequality constraints with their feasibility region, determine the common region of all the constraints including non-negativity restrictions. It is known as common feasible region or area. All the points in this area represent a solution to the problem. This area lies in the first quadrant of the graph because of non-negativity restrictions. For less than or equal to and less than constraints the feasible area is on or below these lines and for greater than or equal to and greater than constraints, the feasible area will be on or above these lines.

(4) The last step is tracing the point from the common feasible region which optimizes the objective function. The optimal solution lies in the corner points. Therefore optimal solution can be evaluated at various corner points.

Illustration 3.

Klunk and Klick is to maximise profits by prod. products and/or product *B*. both of which have to be processed on two machines 1 and 2. Pro A requires 2 hours on both the Machines 1 and 2. while product *B* requires 3 hours on Machine 1 but only 1 hour on Machine 2. There are only 12 and 8 hours available on Machine 1 and 2 respectively. The profit per unit is estimated at ₹ 6 and ₹ 7 in case of A and B respectively. **Follow** the graphical method of linear programming.

Solution

First step. Formulate Linear Programming Problem. The first step is to formulate the linear programming problem by restarting the above information in mathematical form. First, of all. 'objective function' is to be stated. Here this term would refer to an equation snowing the relationship between output and profit. Now if

P = Profit

₹ 6A = Total Profit from sale of Product A

₹ 7*B* = Total Profit from sale of Product *B*

Maximize P = ₹6A + ₹ 7*B*

Subject to following constraints:

$2A + 3B \leq 12$ …(1)

$2A + B \leq 8$ …(2)

$A \geq 0, B \geq 0$

Second Step. Plot the Constraints on Graph. The next step is to plot the constraints of the linear programming problem on a graph; Products is shown on the X-axis (horizontal axis) and Product B is shown on V-axis (vertical axis). This inequality 2A + 3B ≤ 12 may be drawn on the graph by first locating its two terminal points and then joining the points by a straight line. The two terminal points for inequality can be found out in the following manner:

(a) If It is assumed that all the time available on Machine 1 is used for Product/I. it would mean that production of Product *B* would be zero and then 6 units of Product A would be made. Thus ifB = 0. then A < 6. If we produce the maximum number of product, then *A* = 6. So our first point is 6. 0. This point denotes zero production of B and 6 units of *A*.

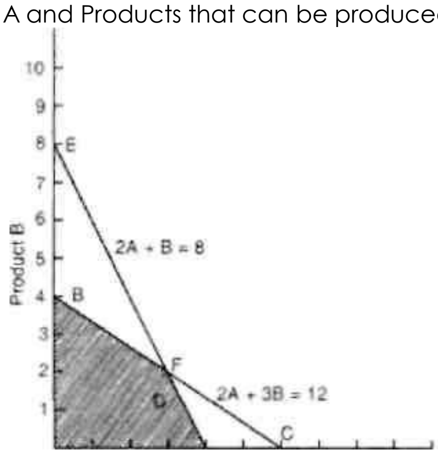
(b) In order to find the second point. It is assumed that all the time available on Machine 1 is used in making Product B. that is. production of A Is zero. Under this assumption maximum number of B will be produced. Then B = 4. So second point is (0. 4). This point denotes output of 4 units of Band zero units of *A*. Locating these two points. *i.e.,* (6, 0) and (0, 4) and joining them. We get a straight line *BC* (see Fig. given below). This line shows the maximum quantities of products *A* and B, that can be produced on Machine 1. The area *BOC* in the graphic representation of the inequality 2*A* + 3B ≤ 12. It is emphasized that the inequality Is represented by the area *BOC* and not the line *BC*.

In the same way. the other inequality 2*A* + *B* ≤8 can also be drawn graphically. For this purpose also, we obtain two points as follows:

(a) If output of B is zero, the maximum output of *A* on Machine 2 will be 4. So first point for this is (4, 0): and

(b) If output *of A* is zero, the maximum output of B on Machine 2 will be 8. Therefore, the second point will be (0,8).

Locating these points and joining them, we get a straight line EF (see Fig. below). This line represents maximum quantities of Product A and Products that can be produced on Machine 2.



Third Step. Identify feasibility region and ascertain coordinates of its, corner points.

Third step to identify the cross shaded area OBDF(in above Fig.) generally known as feasibility region or feasibility area and then ascertain the coordinates of its corner points. As will be seen, the feasibility area is formed with the following boundaries :

x-axis

y-axis

BDF boundary. This is formed by the intersection of lines *BC* and EF at point D. If a point is to satisfy both the constraints and the non-negativity conditions, it must fall inside the cross shaded area on its boundaries. All points outside the feasibility area are inadmissible. For example, if we begin at the origin O, we can travel no further to the right than point *F*. If we were to proceed further, capacity restrictions of Machine 2 will be violated. Likewise, moving from O to B. we cannot proceed beyond B because we do not have more than 12 hours on Machine 1. Similarly moving beyond O leftward or downward woidd not satisfy non-negativity conditions. Having identified the feasibility region, we have to direct our attention to its corner points, because the optimum solution invariably must be on the one of these corner points. We already know the coordinates of three corner points, *viz.,*

O (0, 0)

B (0., 4)

F(F, 0)

The coordinates of point *D*. however, are yet to be ascertained. One method can be to read its coordinates through an accurately drawn graph. Another method is to solve simultaneously the equations of the two lines which intersect to form point D. The equations to be solved are:

$$2A+3B = 12$$
$$\underline{2A + B = 8}$$
$$2B = 4$$
$$\mathbf{B = 2}$$
$$\therefore \quad B = 2$$

Now substituting the value of *B* in the equation 2A + 3B = 12. we get

2A + 6 = 12 *or A* = 3 So, the coordinates of point D are (3,2).

Fourth Step. Test which corner point is most profitable. The fourth step is to test the corner point, *viz., O, B,* D and F of the feasibility region *OBDF* in order to see which corner-point yields the maximum profit. Thus

Corner point O (0 ,0) =6(0) + 7(0) =0

Corner point B (0 ,4) =6(0) + 7(4) = 28

Corner point F (4, 0) = 6(4) + 7(0) = 24

Corner point D (3,2)= 6(3) + 7(2) = 32

The corner point which yields the maximum profits in D. The

So far we have considered problems for which unique optimum solution exist. But, in actual practice linear programming may be such that a unique optimum solution does not exist. These exceptional cases can be, where no feasible solution exist, unbound solution, multiple solution or degenerate solution. Let us illustrate these situations in graph.

(i) No Feasible Solution

In a linear programming problem it may happen that no solution is possible. It arises where constraints are such that they do not form a common feasible region,

Illustration 4

Maximize $Z = 2x_1 + x_2$

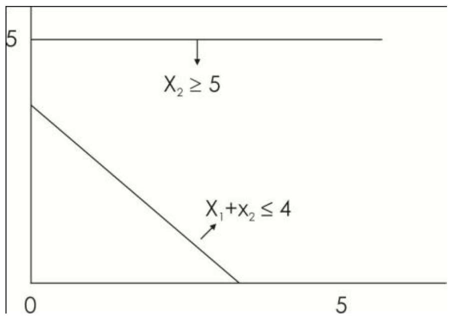
Subject to, $x_1 + x_2 \leq 4$

$x_2 \geq 5$

$x_1 , x_2 \geq 0$

Solution:

By drawing graph, we find



(ii) Unbounded solution

A linear programming problem may have unbounded solution which means it has no limit on the constraints. It simply means the common feasible region is not bounded in any respect. The primary variables can take any value in the unbounded region. In such situations, the optimum solution may exist or may not exist or may not exist. It will not be existing if the value if objective function goes on changing in the unbounded region. But the optimal solution will be existing of the value of objective function in the unbounded feasible region is less than the value at the vertex.

Illustration 5

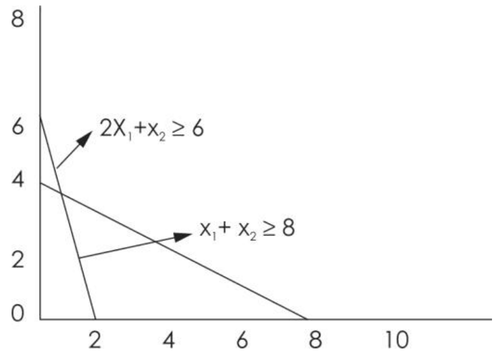
Maximize $P = 2x_1 + 3x_2$

Subject to $2x_1 + x_2 \geq 6$

$x_1 + x_2 \geq 8$

$x_1, x_2 \geq 0$

Solution: By drawing a graph, we find out common feasible region as unbounded.



(iii) Multiple Solution:

Another possible outcome of linear programming problem may be in the form of multiple solution, Its means that there are more than one solution which optimizes the objective function. According to extreme point theorem if the value of objective function is same at more than one vertex of the feasible region, all the points on the bounded of common feasible region represents the optimum solution.

Illustration 6

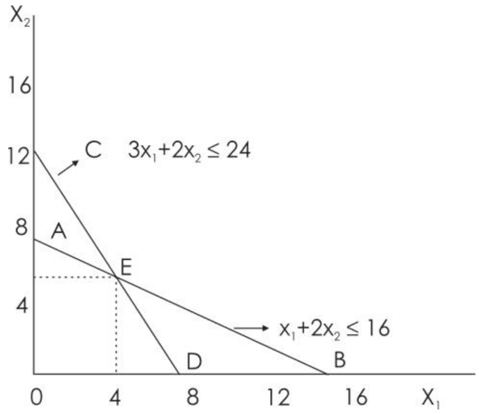
Maximize $Z = 6x_1 + 4x_2$

Subject to

$x_1 + 2x_2 \leq 16$

$3x_1 + 2x_2 \leq 24$

$x_1, x_2 \geq 0$



The optimal solution is at E (4,6) where P =₹48 but at D (8,0) also P = ₹48. Hence it means that all the points at E D boundary line, has got optimum solutions. Therefore the objective function is parallel to second constraint.

(iv) Degeneracy

The linear programming problem may be degenerate type. Degeneracy happens when the two constraint boundary, where one constraint is redundant, intersect at one axis of graph. Redundant constraint is that which is not affecting the feasible region in any way. In this situation, there can be two possibilities either optimal solution exist or does not exist. If it exists, the solution may be degenerate or non-degenerate.

C. Simplex Method

Steps involved in the Simplex Method

1. First step is to formulate the linear programming problem by restating the information in mathematical form. i.e., writing objective function and constraints mathematically.
2. Develop equations from the inequalities by adding slack variables.
3. Develop initial simplex tableau including the initial solution.
4. Calculate Z_j and $C_j - Z_j$ values for this solution.
5. Select the optimum (or pivot) column, i.e., the column with the highest positive number in the C_j and Z_j row.
6. Select the row to be replaced, that is, pivot row, by dividing quantity column values by their corresponding column values and then choosing the smallest non-negative quotient.
7. Compute the values for replacing rows.
8. Compute the value for remaining rows.
9. Calculate Z_j and $C_j - Z_j$ value for this situation.
10. If there is a non-negative $C_j - Z_j$ value, proceed as indicated in Step 5 above.
11. If there is no non-negative $C_j - Z_j$ value, the final solution is reached.

Illustration 7

A firm manufacturers and sells two products Alpha and Beta. Each unit of Alpha requires 1 hour of machining and 2 hours of skilled labour, whereas each unit of Rate uses 2 hours of machining and 1 hour of labour. For the coming month the machine capacity is limited to 720 machine hours and the skilled labour is limited to 780 hours. Not more than 320 units of Alpha can be sold in the market during a month.

(i) Develop a suitable model that will enable determination of the optimal product mix.

(ii) Determine the optimal product-mix and the maximum contribution. Unit contribution from Alpha is ₹6 and from Beta is ₹4.

(iii) What will be the incremental contribution per unit of the machine hour, per unit of labour, per unit of Alpha saleable?

Products	Machining	Skilled Labour	Contribution
Alpha	1 hr	2 hr	6
Beta	2 hr	1 hr	4
Available hours	720 hr	780 hr	

Solution:
Let x_1 be the no. of units of Alpha produced
x_2 be the no. of units of Beta produced.

Objective function:
Max. $Z = 6x_1 + 4x_2$
Subject to constraints
$x_1+2x_2 \leq 720$
$2x_1+x_2 \leq 780$
$x_1 \leq 320$ and
$x_1, x_2 \geq 0$
$x_1+2x_2+ S_1=720$
$2x_1+x_2+ S_2=780$
$x_1+s_3 = 320$
Max. $Z = 6x_1+4x_2 + 0.S_1 + 0.S_2+ 0.S_3$

		6	4	0	0	0	
C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min. Ratio
0	720	1	2	1	0	0	720/1 = 720
0	780	2	1	0	1	0	780/2 = 390
0	320	1	0	0	0	1	320/1 = 320
	0	6	4	0	0	0	

0	400	0	2	1	0	-1	400/2=200
0	140	0	1	0	1	-2	140/1=140
6	320	1	0	0	0	1	320/0=a
	1920	0	-4	0	0	6	

0	120	0	0	1	-2	3	120/3=40
4	140	0	1	0	1	-2	140/-2=-70
6	320	1	0	0	0	1	320/1=320
	2480	0	0	0	4	-2	

0	40	0	0	1/3	-2/3	1
4	220	0	1	2/3	-1/3	0
6	280	1	0	-1/3	2/3	0
	2560	0	0	2/3	8/3	0

∴ $x_1 = 280$
$x_2 = 220$
$Z = 2560$

Primal and Dual

Illustration 8

A Company produces the products P, Q and R from three raw materials A, B and C. One unit of product P requires 2 units of A and 3 units of B. A unit of product Q requires 2 units of B and 5 units of C and one unit of product R requires 3 units of A, 2 unit of B and 4 units of C. The Company has 8 units of material A, 10 units of B and 15 units of C available to it. Profits/unit of products P, Q and R are ₹3, ₹5 and ₹4 respectively.

(a) Formulate the problem mathematically,
(b) How many units of each product should be produced to maximize profit?
(c) Write the Dual problem.

Solution:

Raw Materials	P	Q	R	Available units
A	2	-	3	8
B	3	2	2	10
C	-	5	4	15

Profits are ₹3, ₹5 and ₹4
Let x_1 be the no. of units of P
Let x_2 be the no. of units of Q
Let x_3 be the no. of units of R
Objective function: Max. $Z = 3x_1 + 5x_2 + 4x_3$

Subject to constraints:
$2x_1+3x_2 \leq 8$
$3x_1+2x_2+2x_3 \leq 10$
$5x_2+4x_3 \leq 15$
And $x_1, x_2, x_3 \geq 0$.
Primal
Max.$Z = 3x_1+5x_2+4x_3$
Subject to
$2x_1+3x_2 \leq 8$
$3x_1+2x_2+2x_3 \leq 10$
$5x_2+4x_3 \leq 15$
And $x_1, x_2, x_3 \geq 0$
Dual
Min. $Z = 8y_1+10y_2+15y_3$
Subject to
$2y_1+3y_2 \geq 3$
$3y_1+2y_2+5y_3 \geq 5$
$2y_2+4y_3 \geq 4$
And $y_1, y_2, y_3 \geq 0$

$2x_1+3x_2 + S1= 8$
$3x_1+2x_2+2x_3 + S_2 = 10$
$5x_2+4x_3 + S_3 = 15$

Max $Z = 3x_1+ 5x_2+4x_3+0.S_1+0.S_2+0.S_3$

		3	5	4	0	0	0	
C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	Min. Ratio>0

0	8	2	3	0	1	0	0	8/3=2.67
0	10	3	2	2	0	1	0	10/3=3.33
0	15	0	5	4	0	0	1	15/5=3
	0	-3	-5	-4	0	0	0	

5	8/3	2/3	1	0	1/3	0	0	8/3/0=a
0	14/3	5/3	0	2	-2/3	1	0	14/3/2=7/3
0	5/3	-10/3	0	4	-5/3	0	1	5/3/4=5/12
	40/3	1/3	0	-4	5/3	0	0	

5	8/3	2/3	1	0	1/3	0	0	8/3/2/3=4
0	23/6	10/3	0	0	1/6	1	-1/2	23/6/10/3=23/20
4	5/12	-10/12	0	1	-5/12	0	¼	5/12/-10/12=-1/12
	15	-3	0	0	0	0	1	

5	19/10	0	1	0	3/10	-1/5	1/10	
3	23/20	1	0	0	1/20	3/10	-3/20	
4	11/8	0	0	1	-3/8	¼	1/8	
	2952/160=18.45	0	0	0	3/20	9/10	11/20	

$\therefore x_1 = 23/20$ $x_2 = 19/10$ $x_3 = 11/8$

Z = 18.45